NONLINEAR ANALYSIS OF PULL IN VOLTAGE IN MICRO-CANTILEVER BEAM

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ABSTRACT

Micro-cantilever beams are structures of great interest in MEMS due to their versatility and simplicity to fabricate. The interest in micro-cantilevers has driven investigations from various perspectives including static and dynamic performances under certain influences such as potential fields. This paper is studying the non-linear differential equation that models the dynamics of a microstructure such a cantilever beam which is subjected to electrostatic field. The model is used to evaluate the critical pull-in voltage. This model is analyzed based on the adopted model of stiffness of the cantilever. The one degree of freedom non-linear differential equation used to model the dynamics of the cantilever subjected to electric field close to snap-on is highly stiff and only Isode algorithm was found to yield correct solution to the problem. Isode is equipped with a robust adaptive time step selection mechanism that enables solutions to very stiff problems, as the one under discussion. The equivalent stiffness of the model was considered based on four different models selected from the literature. The stiffness model suitable for the best match in deflection is proved to be different from the model that yields the best match in the resonant frequency. Pull-in voltage is a topic of high interest as most of micro-cantilever like structures operates under electric files. Pull-in voltage has been investigated from the theoretical perspective. Effect of structural damping for large deflection of micro-cantilever beam was studied numerically in this work. Also, effect of different kind of impulse voltages on pulling voltage was studied. A closed from of time response to step voltage for un-damped system was derived and pull in voltage of such system was calculated. At the end, a reduced form of nonlinear ODE that can be used to derive the pull-in voltage is presented.
INTRODUCTION

Micro-cantilever beams are structures of great interest in MEMS due to simplicity to fabricate. Various aspects of static and dynamic behaviours of the micro-cantilever beams subjected to potential fields, thermal effects, manufacturing influence, inter-laminar stress, or geometric configuration have been studied.

Pull-in voltage represents a topic of high interest in the study of micro beams. It is utilized to identify the performance limits of a micro-cantilever beam with specific geometries. Pull-in voltage indicates where the structures become unstable. At this point, the beam is attracted by the fixed electrode when it reaches a position that corresponds to 2/3 of the original gap between the beam and the fixed electrode.

Due to the operation of micro-cantilever under electric fields theoretical and experimental study of pull-in voltage has been conducted in great number of publications. The pull-in voltage is extracted from the governing differential equation which defines the dynamic of the cantilever beam. Energy method [1,2] and Hamiltonian method [3] have been employed to derive the governing equation of deflection for a micro-cantilever beam under electrostatics field. Since the governing equations are nonlinear different approaches have been used to simplify and consequently finding the pull-in voltage. Taylor series are most utilized approach for linearization of the equations [4-8]. Finite Element Method (FEM) is another approach to calculate pull-in voltage of micro-beams [6]. Small deflections were also assumed to determine pull-in voltage and the results of the numerical solution were compared with experimental results [9]. The comparison shows that the longer the cantilever is, the higher the error is. However, the error would be significantly lower when the cantilever is subjected to potentials substantially below of the snap-on voltage [10]. Runge-Kutta algorithm [11] and perturbation method [12] have been employed to solve the Duffing equations derived to model the dynamics of micro-cantilever beam under electrostatic field and harmonic excitation, respectively. Continuum models for small deflection for micro-cantilever beams was studied and Taylor series built as orthogonal functions for linearization of the ODE [13]. Dimensionless continuous beam theory has been utilized to derive the governing equations of dynamic behaviour of the micro-beam [14]. The effect of width and thickness of the beam on the resonant frequency has been experimentally and theoretically investigated in [15].

GOVERNING EQUATIONS

The dynamic behaviour of an electrified micro-beam is modeled by an equivalent lump mass and a spring as shown in Figure 1.
The dynamics of the system in MEMS is defined by the following governing equation as [43]:

$$\frac{d^2 y(t)}{dt^2} + 2\xi \omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \frac{f(t)}{m}$$  (1)

where \(y\) defines the deflection of the beam and \(f(t)\) is the electrostatic force which has been introduced as

$$f(t) = \frac{\varepsilon_0 A V^2}{2(g - y(t))^2}$$  (2)

in which \(\varepsilon_0\), \(A\), \(V\), \(g\) and \(y(t)\) are electrical permittivity of air, cross section area, voltage, gap distance and deflection of the beam respectively. \(\omega_n\) and \(m\) are used to define the natural frequency and mass of the system, respectively. As a result, the equation (1) can be rewritten as:

$$\frac{d^2 y(t)}{dt^2} + 2\xi \omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \frac{\varepsilon_0 A V^2}{2m(g - y(t))^2}$$  (3)

where the initial conditions (initial speed at reference position and time) for this ODE are respectively assumed as follows:

$$y|_{t=0} = 0 \quad \text{and} \quad \frac{dy}{dt}|_{t=0} = v_0$$  (4)

This is a nonlinear equation and so far no analytical solution has been proposed to find its answer in a close form. This equation becomes more complicated when one extends it to micro-level dimensions which may result in a stiff ODE. There have been many contributions in the literature to solve this problem by using numerical methods [16-18]. The main contribution of the current paper is solving equation (3) by using Lie symmetry method. In this method, the order of differential equation is reduced by one order. Therefore, instead of solving a second order ODE, a first order ODE is solved that by all means it is easier to treat. With cumbersome mathematical method which is called Lie symmetric one can show that the governing equation (3) is reduced to

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Fig 1. The schematic of a mass-spring damper system of a beam
\[-\frac{dv}{dr} + 2\xi \omega_n v^2 + (\omega_n^2 r - \frac{\epsilon_0 AV^2}{2m(g - r)^2})v^3 = 0 \]  \tag{5}

in which \[v = \frac{1}{dy} \frac{dt}{dt} \tag{6}\]

This is a first order ODE with \(v(0) = \frac{1}{v_0}\) as the initial condition. According to Lie symmetry \(v = \frac{1}{dy(t)}\) and \(r = y(t)\) so:

\[\frac{1}{dy(t)} = \frac{m(g - y(t))}{\omega^2 y(t)^3 m - \omega^2 y(t)^2 mg + \epsilon avo^2 - c_1my(t) + c_1mg} \tag{9}\]

and

\[t = \int \frac{m(g - y)}{\omega^2 y^3 m - \omega^2 y^2 mg + \epsilon avo^2 - c_1my + c_1mg} dy + c_2 \tag{10}\]

The above integral can be simplified as below:

\[t = \frac{2\sqrt{m} (EllipticF(j, k)(d - g) + EllipticPi(j, h, k)(e - d))}{\sqrt{(c - e)(c - g)}} + c_2 \tag{11}\]
in which

\[
h = \sqrt{\frac{g-e}{g-d}} \quad k = h \times \sqrt{\frac{(c-d)}{(c-e)}} \quad j = \frac{1}{h} \times \sqrt{\frac{e-x}{d-x}}
\]

(12)

and EllipticF is incomplete elliptic integral of the first kind and EllipticPi is incomplete elliptic integrals of the third kind. Parameters c, d and e are defined as the values of X which are the roots of the following equation

\[
\omega^2 X^3 m - \omega^2 X^2 mg + \varepsilon \alpha v^2 - c_i m X + c_i mg = 0
\]

(13)

As an example of the solution a cantilever beam with length of 200 µm, width of 20 µm, thickness of 2 µm and the gap is 8 µm is assumed. The deflection of the beam when it is electrified with pull-in voltage and a voltage which is higher than pull-in voltage is determined and plotted in Figure 2 and 3, respectively.

*Fig 2.* Deflection of the micro cantilever beam versus time when it is electrified with 112.3346 V (pull-in voltage)
Fig 3. Deflection of the micro cantilever beam versus time when it is electrified with 112.3347 V.

As it can be realized the pull-in voltage in this special case is equal to 112.3347 V. For a voltage higher than the pull-in voltage, the beam shows an unstable behaviour which is correlated to the time when the beam snaps on the bottom electrode. It is important to mentioned that when the operating electrical field is less than pull-in voltage the behaviour of the beam is harmonic.

PARAMETRIC STUDY

The behaviour of the micro cantilever beam with defined geometry as presented in previous section under different voltages has been conducted as a parametric study. The effect of the variation of electrical voltage in vicinity of the pull-in voltage, on the dynamic behaviour of the beam has been calculated and the results are presented in Figure 4.

Fig 4. Dynamic behaviour of the beam exposed to different voltages close to pull-in voltage.
As it can be realized from the Figure 4, the magnitude of the applied voltage has a great effect on the dynamic behaviour of the beam. It can be seen that when the voltage is approaching to the pull-in voltage, a flat region is appeared in the plots which is correlated to saddle point of the equation.

CONCLUSION

In the present paper the dynamic behaviour of a micro- cantilever under electrical field with voltages close to pull-in voltage was analytically investigated and validated experimentally against literature data. The Lie symmetry method was employed to reduce the order of the ODE and consequently the particular exact solution of the governing equation was presented. The pull-in voltage of a beam with a defined geometry was calculated. It was shown that the behaviour of the beam significantly depends on the pull-in voltage. Beams show unstable behaviour when are subjected to a fields created by potentials higher than pull-in voltage.

REFERENCES