Ultrafast Wave Finite Element Method for the computation of dispersion properties in periodic viscoelastic waveguides

Antonio Palermo¹, Alessandro Marzani¹

¹ Department of Civil, Chemical, Environmental and Materials Engineering - DICAM, University of Bologna, Viale del Risorgimento 2, 40136, Bologna, antonio.palermo@unibo.it

Keywords: dispersion relation, linear viscoelasticity, attenuation curves, WFEM, CMS

Abstract
In this work a procedure to compute the dispersion curves for one dimensional viscoelastic waveguides exploiting the finite element based global mass matrix and complex stiffness matrix is proposed. The global matrices of a finite length portion of the waveguide, the unit cell, are post-processed by enforcing Bloch-type boundary conditions along the wave propagation direction and next used to formulate a complex \( k(\omega) \) quadratic eigenvalue problem. The roots of the eigenvalue problem at different frequency values \( \omega \) yield in addition to the wavenumber (phase velocity) information also the attenuation dispersion curves for the waveguide. No finite element coding is needed since the global mass and stiffness matrix can be obtained from commercial FE software. To improve the computational efficiency a modal reduction scheme based on the Component Mode Synthesis method has been applied to reduce the dimension of the eigenvalue problem. As a result the computational time is enormously reduced without loss of accuracy in the complex roots calculation.

1. INTRODUCTION
Guided waves are acknowledged as a mean for nondestructive evaluation (NDE) and structural health monitoring (SHM) purposes as they allows for the development of long range inspection techniques. To such purpose the knowledge of the dispersive attenuation properties of guided waves play a pivotal role. For mono-dimensional waveguides, where geometrical spreading is absent, attenuation of guided waves can be due to material absorption and/or energy leakage in the media surrounding the waveguide. A common and well accepted approach for the computation of guide waves attenuation curves is to look for the roots of the dispersive equation in terms of complex wavenumbers \( k = k_{Re} + ik_{Im} \) for given real wave circular frequencies \( \omega \), by solving a so-called \( k(\omega) \) problem, where the real part of the wavenumber represents the propagation constant and the imaginary part is used to represent the attenuation as wave decay in space.

To date several approaches have been proposed to predict the attenuation dispersion curves in waveguides. In particular:

- analytical methods based on the Superposition of Partial Bulk Waves (SPBW) have been used to predict attenuation curves due to material absorption and wave leakage [1–5]; SPBW methods are limited to waveguides with regular cross section (plates or pipes) and suffer in the computation of the roots of the transcendental dispersive relation especially when, as in this case, complex wavenumbers are sought for real frequency;
- a Spectral collocation method has been recently proposed [6, 7] to compute the attenuation curves for viscoelastic composites laminates; this method does not suffer in the computation of complex roots but still is limited to waveguides of standard cross-section;
- Semi Analytical Finite Element (SAFE) methods have been used to predict material absorption in regular and arbitrary cross-section waveguides [8]; SAFE methods leads to a dispersive wave
equation in the form of an eigenvalue problem that can be easily solved via reliable routines; however requiring the development of ad-hoc finite element like formulations they result accessible to some research groups only;

- the Boundary Element Method (BEM) has been used to compute the attenuation curves for irregular cross-section waveguides [9]; as for SAFE based approaches, BE methods describe the dispersive wave equation as an eigenvalue problem, but they need the development of the dedicated formulation and for this reason are scarcely adopted;
- SAFE method coupled to other schemes like (i) the Superposition of Partial Bulk Waves (SPBW) [10], (ii) the Boundary Element Method (BEM) [11], (iii) the Perfectly-Matched Layer (PML) [12,13], (iv) semi infinite finite elements [14] or with (v) particular boundary conditions BCs [15], have been proposed to predict guided wave leakage; also in these cases special and dedicated formulations have been developed.

In addition to the above mentioned methods, approaches based on standard finite elements, or even finite element software products, generally known as wave finite element methods (WFE), exist [16–18]. WFE methods exploit a FEM model of a finite length portion of the waveguide, the unit cell, and impose Floquet-Bloch boundary conditions at the edges of the cell to build the dispersive wave equation. Such relation results in the form of a generalized eigenvalue problem that can be solved with standard routines. Since the methods rely on available libraries and tools, or even commercial softwares for finite elements, they do not require the development of particular finite elements (as for SAFE and/or BEM schemes) resulting thus widely accessible. Here, we apply this methodology to extract dispersion and attenuation curve of linear viscoelastic waveguides. To this aim first linear viscoelastic constitutive relations are introduced at the finite element level to model material damping. The so-called hysteretic frequency independent rheological model is considered. The introduction of linear viscoelasticity does not require any specific implementation effort since it only requires to use complex moduli for the materials in standard FEM frameworks for linear elasticity. The WFE procedure is applied to form a generalized eigenvalue problem in the complex wavenumber as unknown. The eigenproblem is linearized and the roots of the \( k(\omega) \) dispersion relation are found for real frequency in input allowing thus to trace both wavenumber \( k_{Re} \) and attenuation \( k_{Im} \) dispersion curves.

Nonetheless when complex 3D waveguides are of interest, the computational effort required for dispersion curves extraction can become relevant, limiting the suitability of the method. To this aim, a model reduction technique based on the Component Mode Synthesis method recently proposed for linear elastic waveguides [19] is applied. The Component Mode Synthesis (CMS) method is based on a modal reduction of the internal degree of freedoms (DOFs) of the unit cell, leaving the boundary DOFs untouched and thus available for imposing Floquet-Bloch boundary conditions. The reduction base is built by considering the mode shapes of the unit cell with boundary DOFs restrained, also termed fixed-interface modes. As a result the CMS allows to reduce the model dimensions and so to speed up the dispersion curve computation.

In this work, a viscoelastic steel bar with octagonal cross-section is considered as a representative of complex cross-section damped waveguides. Using the SAFE results for the same problem as reference solution, times and accuracy in the computation of the complex roots via the proposed method are evaluated for different numbers of the fixed interface modes.

2. WAVE FINITE ELEMENT (WFE) METHOD FOR VISCOELASTIC WAVEGUIDES

The starting point of the procedure is the identification of the waveguide unit cell, obtained by cutting a portion of the waveguide of length \( L \) along the longitudinal axis \( z \). The unit cell is then meshed with a standard FE software (we used COMSOL® for all the analyses) ensuring an identical discretization at the left and right boundaries of the cell, later required for one-to-one mapping of the boundary nodes. Material damping is introduced by means of linear viscoelasticity. The beauty of linear viscoelasticity is that, in force of the correspondence principle, it can be introduced by simply replacing the linear elastic
coefficients with their corresponding linear viscoelastic. In particular, when hysteretic material damping model is assumed, the complex Young’s modulus $E^{*}$ and the complex Poisson ratio $\nu^{*}$ easily derivable from the material bulk speeds and attenuations [20], can be used. Standard FE assembly procedure leads to the definition of the unit cell equation of motion:

$$(K^{*} - \omega^2 M)v = f$$  \hspace{1cm} (1)$$

with $K^{*}$ and $M$ the complex stiffness matrix and the mass matrix, respectively, $v$ the vector of nodal displacements and $f$ the vector of nodal forces. By introducing the subsets of nodal DOFs belonging to (i) the left edge of the unit cell ($n_L$), (ii) the right edge of the unit cell ($n_R$) and (iii) the interior DOFs ($n_i$), the discrete operators in the unit cell equation of motion can be partitioned as follows:

$$
\begin{pmatrix}
K_{LL} & K_{LR} & K_{Li} \\
K_{RL} & K_{RR} & K_{Ri} \\
K_{iL} & K_{iR} & K_{ii}
\end{pmatrix}
- \omega^2
\begin{pmatrix}
M_{LL} & M_{LR} & M_{Li} \\
M_{RL} & M_{RR} & M_{Ri} \\
M_{iL} & M_{iR} & M_{ii}
\end{pmatrix}
\begin{bmatrix}
v_L \\
v_R \\
v_i
\end{bmatrix}
= \begin{bmatrix}
f_L \\
f_R \\
0
\end{bmatrix}
$$  \hspace{1cm} (2)$$

For free waves propagating along the waveguide, displacements and forces at successive cross sections can be related as:

$$
\begin{bmatrix}
v_R \\
f_R \\
v_i
\end{bmatrix}
= \lambda
\begin{bmatrix}
v_L \\
f_L \\
0
\end{bmatrix}
$$  \hspace{1cm} (4)$$

where $\lambda = e^{ikL}$ is the Floquet-Bloch propagator and $k = k_{Re} + ik_{Im}$ the complex wavenumber. The Floquet-Bloch operators $\Lambda$ and $\bar{\Lambda}$ can be introduced, respectively, to impose the periodicity of the displacements along successive cross sections $v = \Lambda v_{red}$ [21]:

$$
\begin{bmatrix}
v_L \\
v_R \\
v_i
\end{bmatrix}
= \begin{bmatrix}
I_L \\
I_L\lambda \\
0
\end{bmatrix}
\begin{bmatrix}
v_L \\
v_i
\end{bmatrix}
$$  \hspace{1cm} (5)$$

and to enforce the equilibrium of the unit cell $\bar{\Lambda}^T f = 0$:

$$
\begin{pmatrix}
I_L & I_L\lambda^{-1} & 0 \\
0 & 0 & I_i
\end{pmatrix}
\begin{bmatrix}
f_L \\
f_R \\
0
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
$$  \hspace{1cm} (6)$$

where the reduced set of nodal displacements $v_{red}$ reads:

$$
v_{red} = \begin{bmatrix}
v_L \\
v_i
\end{bmatrix}
$$  \hspace{1cm} (7)$$

In both the operators $\Lambda$ and $\bar{\Lambda}$ the identity matrices $I_L$ and $I_i$ have dimension $(n_L \times n_L)$ and $(n_i \times n_i)$, respectively. Pre- and post- multiplying the unit cell equation of motion with the Bloch operators $\bar{\Lambda}^T$ and $\Lambda$, leads to eigenvalue problem:

$$
\bar{\Lambda}^T (\lambda^{-1}) D(\omega) \Lambda(\lambda) v_{red} = 0
$$  \hspace{1cm} (8)$$

that can be rearranged in the following quadratic form:

$$
[D_0 + \lambda D_1 + \lambda^2 D_2] v_{red} = 0
$$  \hspace{1cm} (9)$$
where:

\[ D_0 = \begin{bmatrix} D_{RL} & D_{Ri} \\ 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} D_{LL} + D_{RR} & D_{Li} \\ D_{IL} & D_{ri} \end{bmatrix}, \quad D_2 = \begin{bmatrix} D_{LR} & 0 \\ D_{iR} & 0 \end{bmatrix} \] (10)

Solution of the eigen problem for different frequencies \( \omega \) provides the waveguide complex dispersion relation \( k(\omega) \).

Despite the ease of implementation, the above WFE method suitable to deal with viscoelastic waveguides requires high computational costs and can suffer of numerical issues due to dynamic matrix ill-conditioning when large number of DOFs are considered. Indeed, high computational cost remains a major drawback of all WFE methodologies when 3D unit cells are analyzed. To this reason, in the following section, the proposed WFEM formulation is coupled with a model reduction approach based on the Component Mode Synthesis (CMS) method.

### 3. CRAIG BAMPTON REDUCTION FOR VISCOELASTIC WAVEGUIDES: FAST WFE METHOD

In this section the Component Mode Synthesis method with a Craig Bampton (CB) reduction scheme is proposed as a mean to reduce the dimension of the \( k(\omega) \) eigenvalue problem. The combination of a WFE approach with a CMS reduction has been originally proposed by Zhou et al. [19] for periodic elastic waveguides and is here detailed and validated for viscoelastic waveguides. In particular we combine the CB reduction with the Bloch operator approach for fast extraction of dispersion curves. To this aim, the displacements of the discretized unit cell are partitioned in internal (\( i \)) and boundaries (\( b \)) nodes:

\[ v = \begin{bmatrix} v_i \\ v_b \end{bmatrix}, \quad \text{with} \quad v_b = \begin{bmatrix} v_L \\ v_R \end{bmatrix} \] (11)

Accordingly, the complex stiffness \( K^s \) matrix and mass matrix \( M \) are partitioned in the sub-matrices:

\[ K^s = \begin{bmatrix} K_{ii}^s & K_{ib}^s \\ K_{bi}^s & K_{bb}^s \end{bmatrix}, \quad M = \begin{bmatrix} M_{ii} & M_{ib} \\ M_{bi} & M_{bb} \end{bmatrix} \] (12)

The Craig Bampton transformation operator \( T_{CB} \) is introduced as:

\[ v = T_{CB} v_{CB} \quad \begin{bmatrix} v_i \\ v_b \end{bmatrix} = \begin{bmatrix} \Phi_i \\ \Psi_{ib} \end{bmatrix} \begin{bmatrix} \eta \\ v_b \end{bmatrix} \] (13)

where \( \Phi_i \) is a set of \( n_{fi} \) fixed-interface modes \( \phi_i \), i.e. the eigenvectors \( \phi_i \) associated to the first smallest \( n_{fi} \) eigenfrequencies of the eigen problem:

\[ [K_{ii}^s - \omega^2 M_{ii}] \phi_i = 0 \] (14)

and \( \Psi_{ib} = -K_{bi}^{-1} K_{ib}^s \) are the set of constrained boundary modes, i.e. static deformations of the system resulting from the application of a unit displacement to one boundary dof, with all other boundary dofs restrained. As such, the interior nodes are substituted by a reduced number of modal \( \eta \) coordinates \( n_{fi} < < n_i \), while boundary \( v_b \) displacements are left untouched. Accordingly, the reduced stiffness and mass matrices for the unit cell, \( K_{CB}^s \) and \( M_{CB} \), respectively, are obtained as:

\[ K_{CB}^s = T_{CB}^T K^s T_{CB}, \quad M_{CB} = T_{CB}^T M T_{CB} \] (15)

and from the reduced operators \( K_{CB}^s \) and \( M_{CB} \), the reduced dynamic stiffness matrix is assembled as:

\[ \mathbf{D}_{CB} = \begin{bmatrix} D_{LL} & D_{LR} & D_{Lf}\i \\ D_{RL} & D_{RR} & D_{Rf}\i \\ D_{fL} & D_{fR} & D_{f\i} \end{bmatrix} \] (16)
Introducing the reduced Bloch operators $\Lambda_{CB}$ and $\Lambda_{CB}^T$:

$$
\Lambda_{CB} = \begin{bmatrix}
I_L & 0 \\
I_L \lambda & 0 \\
0 & I_{fi}
\end{bmatrix}
$$

$$
\Lambda_{CB}^T = \begin{bmatrix}
I_L & I_L \lambda^{-1} & 0 \\
0 & 0 & I_{fi}
\end{bmatrix}
$$

(17)

and pre- and post multiplying the reduced dynamic stiffness matrix $D_{CB}$ by the reduced Bloch operators $\Lambda_{CB}^T$ and $\Lambda_{CB}$, respectively, leads to a reduced quadratic eigenvalue problem as:

$$
[D_{0,CB} + \lambda D_{1,CB} + \lambda^2 D_{2,CB}] \nu_{\text{red},CB} = 0 \quad \text{with} \quad \nu_{\text{red},CB} = \begin{bmatrix}
\nu_L \\
\eta
\end{bmatrix}
$$

(18)

where:

$$
D_{0,CB} = \begin{bmatrix}
D_{RL} & D_{Rfi} \\
0 & 0
\end{bmatrix}
$$

$$
D_{1,CB} = \begin{bmatrix}
D_{LL} + D_{RR} & D_{Lfi} \\
D_{fli} & D_{fri}
\end{bmatrix}
$$

$$
D_{2,CB} = \begin{bmatrix}
D_{LR} & 0 \\
D_{fri} & 0
\end{bmatrix}
$$

(19)

Solution of the reduced eigenvalue problem of eq. (18) at the given frequency step $\omega$ provides the waveguide dispersion curves. Moreover, the reduced block modal shapes $\nu_{R,\text{CB}}$ can be expanded to reconstruct the full modal shape of the unit cell waveguide:

$$
\nu = T_{CB}^T \Lambda_{CB} \nu_{\text{red},CB}
$$

(20)

4. NUMERICAL APPLICATIONS

The proposed framework is applied to compute the dispersion wavenumber and attenuation curves of an octagonal steel bar, whose mechanical properties are given in Table 1. To this aim, a finite length bar with octagonal cross-section has been modeled in COMSOL® (see Fig. 1(a)). The length of the bar has been taken equal to $a = 0.0075$ m in order to map real wavenumbers up to $\pi/a = 418.88$ [1/m]. The three dimensional mesh has 4064 tetrahedral elements with 10 node per element for a total of 17739 dofs, with 486 boundary dofs, and considering at least 10 nodes for the smallest wavelength $\lambda = a/2$.

The dispersion curves in terms of phase velocity $c_{ph} = \omega/k_{Re}$ and attenuation $att = k_{Im}$ computed for the full model (17739 dofs) and for a CB reduced model with 80 fixed interface modes (for a total of 486 + 80 = 566 boundary-modal coordinates) are represented in Fig. 1(c) and Fig. 1(d), respectively.

In the considered $0 - f_{max} = 180$ kHz frequency range the first 80 fixed interface modes span the frequency range up to $f_{mode} = 3 \times f_{max}$ kHz. This selection criteria ensures accurate results (<2% of discrepancy with full model prediction) for linear elastic waveguides [19]. The obtained curves are compared with those obtained using a semi-analytical finite element (SAFE) formulation, similar to that proposed in Ref. [8]. The SAFE mesh, represented in Fig. 1(b), has been built using linear strain triangular elements with the nodes of the tetrahedral elements belonging to the cross-section surface of the 3D unit cell mesh in Fig. 1(a). As it can be noted, for both the full and reduced model the phase velocity $c_{ph}$ and attenuation $att$ curves perfectly match with those computed by using a SAFE scheme.

As regards to the computational time saving and the accuracy of the method, the reduced model with 80 fixed interface modes requires just the $1.9\%$ of the full model computational time, with an error in the real and imaginary wavenumber component prediction below the 2% (see details in Table 2). In particular, the error in real and imaginary wavenumber components are calculated as:

$$
\text{err}_{k_{Re}} = \max \left| \frac{k_{\text{RED},Re} - k_{\text{FULL},Re}}{k_{\text{FULL},Re}} \right| \quad \text{err}_{k_{Im}} = \max \left| \frac{k_{\text{RED},Im} - k_{\text{FULL},Im}}{k_{\text{FULL},Im}} \right|
$$

(21)

<table>
<thead>
<tr>
<th>Material</th>
<th>$\rho$ [kg/m$^3$]</th>
<th>$c_L$ [m/s]</th>
<th>$c_T$ [m/s]</th>
<th>$\alpha_l$ [Np/Å]</th>
<th>$\alpha_T$ [Np/Å]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel</td>
<td>7800</td>
<td>5960</td>
<td>3190</td>
<td>0.003</td>
<td>0.043</td>
</tr>
</tbody>
</table>

Table 1 : Mechanical parameters.
Figure 1: a) Mesh of the finite length portion of the octagonal waveguide built in COMSOL. b) Cross-section mesh used in the SAFE formulation. c) Phase velocity and d) attenuation dispersion curves.

where the superscripts \textit{FULL} and \textit{RED} refer to the full and reduced model solutions, respectively, and $k_j, Re(Im)$ is the real (imaginary) wavenumber component of the $j$ branch intercepted at the given frequency $\omega$. Nonetheless higher accuracy can be easily achieved by enriching the basis of the CB reduction. For example, considering 500 fixed interface modes the error in the prediction of both imaginary and real wavenumber components can be reduced below 0.2% with a small increase in the required computational time (i.e. 5% of the full model computational time). Moreover the accuracy of the method shows a clear frequency dependence, with lower frequencies characterized by smaller errors as shown in Fig.2(a) and Fig.2(b). This trend is consistent with the criteria used for the modes selection.

5. CONCLUSIONS

In this work the WFE method combined with CMS model reduction has been extended for the computation of the dispersion curves in terms of attenuation for mono-dimensional linear viscoelastic waveguides. To this aim the complex stiffness matrix and the mass matrix of the unit cell, which have been
simply derived from a commercial software, are considered. To reduce the computational cost of the approach, the Craig Bampton reduction scheme has been applied. The results prove that accurate roots (error below the 2%) can be obtained with a computational time two order of magnitude smaller than the one needed to solve the full model, i.e. 0.8 seconds compared to 77 seconds to solve the eigenvalue problem at a given frequency for the considered problem.

REFERENCES


