Integration of Governing Differential Equations of Continuum Mechanics with Scanning Laser Vibrometry Technology for Monitoring of Structural Damage

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Outline

- Introduction
- Proposed Technique
- PSV 3D scanning laser vibrometer
- Smoothing and Differentiation Algorithm
- Experimentation
- Theoretical Study
- Experimental Results
- Conclusion
Introduction

- Currently, there are many damage detection techniques implemented or under development.

- All damage detection techniques are based on a physical or mechanical principle or utilise phenomena accompanying the damage evolution.

- However, the use of governing differential equations of continuum mechanics have been surprisingly over looked (eg. Strain Compatibility and Condition of Equilibrium).
Proposed Techniques

- The technique:
  - is based on the use of two principles from continuum mechanics
    - Strain Compatibility
    - Conditions of Equilibrium
  - detects localised damage by determining violations in these equations from surface displacement and strain fields
  - can be used to detect damage in beam, plate and shell structures, with in-plane or out-of-plane loading
Proposed Techniques

- This seminar will focus on the out-of-plane loading scenario.

- The following slides will discuss
  - the theoretical foundation of technique
  - the measurement system utilised for the technique
  - the filtering and differentiation algorithm
Strain Compatibility

- Strain compatibility has a mathematical and physical meaning.

Image adapted from Sanford 2003
Strain Compatibility

- There are six partial differential equations of compatibility for 3D bodies first derived by Saint-Venant, 1860s.
- For plates components these reduces to one:

  \[
  \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2 \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}
  \]

  \( \varepsilon \) - normal strain,
  \( \gamma \) - shear strain
  \( x, y \) - Cartesian coordinates
Conditions of Equilibrium

- Equilibrium of moments about the x axis and y-axis are, respectively,
  \[
  \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0
  \]
  \[
  \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0
  \]

- Equilibrium of forces in the z-direction is governed by
  \[
  \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0
  \]
Conditions of Equilibrium

- Combining these conditions of equilibrium yields the **differential equation of equilibrium** for bending of thin plates, as below

\[
\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p
\]

- Alternatively, this equation can be represented in terms of stress components \((\sigma_x, \sigma_y, \text{ and } \tau_{xy})\),

\[
\frac{\partial^2 \sigma_x}{\partial x^2} + 2 \frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_y}{\partial y^2} = -p
\]
Conditions of Equilibrium

- Incorporating Orthotropic Plane Stress Hooke’s Law, this equation can be rearranged in terms of strain components

\[
\frac{\partial^2 (E_x \varepsilon_x + E_{xy} \varepsilon_{xy})}{\partial x^2} + 2G \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} + \frac{\partial^2 (E_y \varepsilon_y + E_{xy} \varepsilon_x)}{\partial y^2} = -p
\]

where

\[
E_x = \frac{E'_x}{1 - \nu_x \nu_y} \quad E_y = \frac{E'_y}{1 - \nu_x \nu_y} \quad E_{xy} = \frac{E'_{xy}}{1 - \nu_x \nu_y} = \frac{E'_x \nu_y}{1 - \nu_x \nu_y} = \frac{E'_y \nu_x}{1 - \nu_x \nu_y}
\]

\(G\) is the shear modulus
\(E'_x\) and \(E'_y\) are the Moduli of elasticity
\(\nu_x\) and \(\nu_y\) are Poisson’s ratio
Governing Differential Equations

- Equating the shear strain components of the strain compatibility and equilibrium equations and assuming no lateral loading \((p = 0)\),

\[
E_x \frac{\partial^2 \varepsilon_x}{\partial x^2} + \left( E_{xy} + 2G \right) \left( \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} \right) + E_y \frac{\partial^2 \varepsilon_y}{\partial y^2} = 0
\]

- Alternatively the governing differential equation for out-of-plane displacement \((w)\) is

\[
E_x \frac{\partial^4 w}{\partial x^4} + 2 \left( E_{xy} + 2G \right) \frac{\partial^4 w}{\partial x^2 \partial y^2} + E_y \frac{\partial^4 w}{\partial y^4} = 0
\]

*governing differential equation for deflection of orthotropic thin plates*
Governing Differential Equations

- Interestingly, the previous equations for an isotropic plate reduces to

\[ \nabla^2 (\varepsilon_x + \varepsilon_y) = 0 \]

\[ \nabla^4 w = 0 \]

- For a slender beam

\[ \frac{\partial^2 \varepsilon_x}{\partial x^2} = 0 \]

\[ \frac{\partial^4 w}{\partial x^4} = 0 \]
Orthogonal transformation

Fault location

Specimen showing measurement grid used by PSV

Displacement measured in direction of laser beams

Electromagnetic shakers

Displacement in the three orthogonal axes

Control Box

Displacement measured in direction of laser beams
Numerical Differentiation

- To evaluate the governing differential equations from the measured out-of-plane displacement many numerical differentiation methods could be utilised.

- The results discussed in this seminar utilise a Savitzky–Golay Filter.
Numerical Differentiation

- The main advantage of the Savitzky–Golay Filter is that it generally preserves features of the distribution such as relative maxima, minima and width.

- The Savitzky-Golay filter performs a local polynomial regression (of degree $r$) on a distribution (2$m$+1) to determine the smoothed value of the $n^{th}$ derivative for each point.
Savitzky–Golay Filter

Filter Width \((2m+1 = 5)\)

Evenly Spaced Data Points \((y_i)\)

- Consider a sample containing \(q\) evenly spaced data points \((y_i)\), which is required to be smoothed or differentiated (to order \(r\)) with a \(2m+1\) point filter and polynomial order \(n\).
Savitzky–Golay Filter

Temporary Coordinate System

\[ i = -2 \quad -1 \quad 0 \quad 1 \quad 2 \]

- Requires repeatedly fitting a polynomial of order \( n \) to \( 2m+1 \) consecutive points

- Each group of points is converted to a temporary coordinate system
Savitzky–Golay Filter

- An evaluation point $t$ is utilised to process the data at the edges of the sample.
Savitzky–Golay Filter

- The one dimensional Savitzky-Golay differentiator has the form [8],

\[ f_n^{r}(t) = \sum_{i=-m}^{m} h_{t,i}^{n,r} y_i \]

\[ \Delta x^r \]

- where the convolution weight is given by

\[ h_{t,i}^{n,r} = \sum_{k=0}^{n} \frac{(2k+1)(2m)^k}{(2m+k+1)^{(k+1)}} P_k^{m,0}(i) P_k^{m,r}(t) \]

- the Gram polynomial is given by

\[ P_k^{m,r}(i) = \frac{2(2k-1)}{k(2m-k+1)} \left[ i P_{k-1}^{m,0}(i) + r P_{k-1}^{m,r-1}(i) \right] - \frac{(k-1)(2m+k)}{k(2m-k+1)} P_{k-2}^{m,r}(i) \]

- with \( P_0^{m,r}(i) = 0 \) and \( P_{-1}^{m,r}(i) = 0 \)
Savitzky–Golay Filter

\[ w \xrightarrow{\text{SG Filter}} 3^\text{rd} \text{ Order Polynomial} \xrightarrow{\text{0}^\text{th} \text{ Derivative}} w^* \xrightarrow{\text{SG Filter}} 4^\text{th} \text{ Order Polynomial} \xrightarrow{4^\text{th} \text{ Derivative}} \frac{\partial^4 w}{\partial x^4} \]
Experimentation

- The composite beam was made of 0/90 straight woven 6 ounce E-glass fibre cloth and vinyl ester resin matrix.
- Specimen was manufactured using a wet lay-up technique and left to cure at room temperature.
- Laid-up with 8 layers of cloth producing a 2.9mm thickness.
- The delamination was introduced by placing two strips of Mylar polyester film between the middle layers.
Experimentation

- A 4Hz sinusoid excitation was applied to the end of the cantilever beam via an electromagnetic shaker.

- A line of 501 measurement points \((2m+1)\) was used, spanning a length of 467.8mm along the central line of the beam (0.94 mm spacing).

- Signal to noise ratio of the out-of-plane displacements was measured at 60 dB.
Theoretical Study

Physical model of the delaminated cantilever beam.

- $F = 39 \text{ mN}$
- $\frac{E_2 I_2}{E_1 I_1} = 1.4$

Approximated model of the delaminated cantilever beam.
Theoretical Results - No Noise

\[ \frac{\partial^4 \omega}{\partial x^4} (\text{m}^{-3}) \]

- residual governing differential equation of out-of-plane displacement.
- beam curvature
- location of the delaminated section.

\[ 2m + 1 = 21 \]

Curvature (m\(^{-1}\))

X Position (m)
Theoretical Results - No Noise

\[ 2m + 1 = 41 \]

- residual governing differential equation of out-of-plane displacement.
- beam curvature
- location of the delaminated section.
Theoretical Results - No Noise

![Graph showing the residual governing differential equation of out-of-plane displacement, beam curvature, and the location of the delaminated section.]

2m+1 = 61

- residual governing differential equation of out-of-plane displacement.
- beam curvature
- location of the delaminated section.
Theoretical Results - No Noise

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- residual governing differential equation of out-of-plane displacement.
- beam curvature
- location of the delaminated section.
Theoretical Results - No Noise

![Graph showing residual governing differential equation of out-of-plane displacement and beam curvature with the location of the delaminated section highlighted.](image_url)

- residual governing differential equation of out-of-plane displacement.
- beam curvature
- location of the delaminated section.

\[ 2m+1 = 101 \]
Theoretical Results - No Noise

\[ 2m+1 = 121 \]

- residual governing differential equation of out-of-plane displacement.
- beam curvature
- location of the delaminated section.
Theoretical Results - 60dB Noise

- The governing differential equation for out-of-plane displacement was then evaluated using the simulated deflection data in the presence of noise.

- an applied signal to noise ratio equal to that of the measured beam deflection (SNR = 60dB).

- These results will be compared with the experimental results.
Theoretical Results - 60dB Noise

2m+1 = 61

- residual governing differential equation of out-of-plane displacement.
- beam curvature
- location of the delaminated section.
Theoretical Results - 60dB Noise

2m+1 = 81

- residual governing differential equation of out-of-plane displacement.
- beam curvature
- location of the delaminated section.
Theoretical Results - 60dB Noise

\[ 2m + 1 = 101 \]

- residual governing differential equation of out-of-plane displacement.
- beam curvature
- location of the delaminated section.
Theoretical Results - 60dB Noise

2m+1 = 121

- residual governing differential equation of out-of-plane displacement.
- beam curvature
- location of the delaminated section.
Experimental Results
Experimental Results

![Graph showing experimental results with the following legend:

- **black line**: residual governing differential equation of out-of-plane displacement.
- **gray line**: beam curvature.
- **gray shaded area**: location of the delaminated section.]

The graph shows a plot of curvature ($\theta$) against position ($x$) with markers at specific positions.

- $2m+1 = 61$
Experimental Results

\[ 2m + 1 = 81 \]

- \( \frac{\partial^4 u}{\partial x^4} \) (m\(^{-3}\))
- Curvature (m\(^{-1}\))
- X Position (mm)

- Black line: residual governing differential equation of out-of-plane displacement.
- Gray line: beam curvature
- Shaded area: location of the delaminated section.
Experimental Results

\[ 2m+1 = 101 \]

- residual governing differential equation of out-of-plane displacement.
- beam curvature
- location of the delaminated section.
Experimental Results

2m + 1 = 121

- residual governing differential equation of out-of-plane displacement.
- beam curvature
- location of the delaminated section.
Conclusion

- Many defect detection methods could be developed based on this proposed concept.

- Common features of this method
  - Simplicity
  - work for local and large areas
  - effective for:
    - plate-like, shell-like and composite structures
    - elastic and plastic deformations
    - complex geometries
Future Work

- Investigate the implementation of technique for:
  - delaminated composite plates (out-of-plane loading)
  - cracks propagating from notches, holes, etc (in-plane loading).
  - mixed loading scenarios
Thank you for your attention