Phase velocity method for computing dispersion curves in thin plates

M. Cruz Rodriguez 1, E. Moreno Hernández 1, V. Hernández Mederos 1, J. Estrada Sarlabous 1, A. Mansur Graverán 1

1 Institute of Cybernetics, Mathematics and Physics; Havana, Cuba; Phone: +53 58036497
e-mails: manuel@icimaf.cu, moreno@icimaf.cu, vicky@icimaf.cu, jestrada@icimaf.cu, ahmed@icimaf.cu

Abstract

Lamb waves are extensively used in non-destructive tests (NDT) in thin plates. Their phase and group velocities depend on the frequency and on thickness of the guide plate. In this work, we use the phase velocity method (PVM), in combination with finite element method (FEM), to compute the dispersion curve for phase velocity of an ultrasonic pulse traveling in a thin transversally isotropic plate. The FEM-PVM is based on the numerical solution of the wave propagation equations for several selected frequencies. For each fixed value of the frequency we solve the corresponding partial differential equations with FreeFem++ software. The phase velocity for a given frequency is obtained from the computed displacements at few points on the top of the plate. Dispersion curves are computed for transversally isotropic plates with constant thickness and also for plates with slowly varying thickness.

Keywords: Lamb wave, plate element, NDT, finite element method, phase velocity method, dispersion curve.

1 Introduction

In today’s industry, flat elements are present in many areas such as energy and petrochemical industries, in the inspection of railway tracks, building structures, pipelines of gases and liquids and in the study of aeronautical and aerospace structures [1, 2, 3]. In these fields of industry the good structural state of its components is very important since a premature failure can be very expensive or disastrous. In order to avoid these failures, often non-destructive testing (NDT) are used in component inspections [4].

Within the field of NDTs, Lamb waves are an area of great interest today. These waves propagate long distances in planar structures [5], because they use the structure itself as a waveguide, hence they are suitable to inspect hidden or difficult to access areas, such as partially buried structures covered with protective or insulating material or structures hidden behind other elements. Lamb waves are also known as shear-vertical waves. For isotropic and homogeneous plates one can find two types of Lamb waves: the symmetric and the antisymmetric ones.

In order to carry out these inspections properly, it is necessary to study their propagation along the structures to be evaluated and how they interact with possible defects. This phenomenon is known as geometric dispersion and is characterized by the geometric dispersion curves [6], which describe the relationship between phase velocity (and therefore group velocity) and frequency. This curve is indispensable in the calibration of ultrasonic equipments to study the position and size of a defect. In fact, international standards require knowledge of these techniques.

Dispersion curves can be obtained analytically when the material has a simple geometry. This is for example the case of bars and plates, however in the case of irregular geometries there are
no theoretical models. On the other hand, there is also the problem of the interaction of these waves with various types of defects through reflection and/or transmission mechanisms, where some analytical models may also exist but only for simple geometries.

1.1 Related works

The numerical simulation of guided Lamb wave propagation in particle-reinforced composites is considered in [7]. The finite element method is used to perform parameter studies in order to better understand how the propagation of the Lamb waves in these plates is affected by changes in the central frequency of the excitation signal. The Spectral Finite Element Method, the hierarchical p-FEM and the IgA approach are compared in [8], where they are used to compute the time-of-flight of Lamb waves propagating along a plate of finite length.

Phase velocity method was introduced initially as an experimental method to evaluate specimens of constant thickness. In [9, 10] it was used with continuous excitation. Later, it was extended and validated for pulses in [3, 9, 10, 11]. It was also applied to evaluate experimentally laminations in aircrafts using ultrasonic pulses, due to the strong dependence of the velocity on thickness [11].

For plates of constant thickness, at a given frequency, each propagating Lamb wave has a constant phase velocity and a constant wavenumber. On the other hand, in modern industry it is frequently required to test planar waveguides with continuous variation of thickness [12]. When the plate is of slowly varying thickness, guided waves are called adiabatic modes [13]. Quoting [14], “an adiabatic mode adapts to the thickness variation of the plate: its phase velocity and wavenumber change continuously during propagation, depending on the local thickness. This means that one can consider that locally, the varying thickness plate can be considered as a plate with constant thickness, which is the local thickness of observation”.

The propagation of adiabatic modes in an elastic plate with a slowly linearly varying thickness is considered in [14]. When the waves propagate in direction to decreasing thickness it becomes apparent, in both the experiments and the simulations, that a critical thickness could be reached corresponding to their cut-off. Reflection and conversion phenomena have then been observed. Adiabatic symmetric modes propagating in tapered plates are studied in [12]. The wavenumbers are determined based on the longitudinal displacements in the symmetry plane. A modified signal processing approach adapted to the measurement of guided wave propagation in waveguide of variable thickness is discussed in [15]. The method is based on an equation describing the evolution of the guided modes wavenumbers with respect to position along the direction of propagation in the waveguide and it is validated using experimental data.

1.2 Our contribution

The main contribution of this work is the use of a combination of phase velocity method with finite element method, for the numerical computation of the phase velocity dispersion curve of an ultrasonic pulse traveling in a thin plate. The paper is focused on the case of thin transversally isotropic plates of composite materials and it can be extended to guided modes in waveguides with slowly varying thickness. A FreeFem++ in-house code for computing approximately the
pulse displacement and the plate deformation has been implemented and it is available at [16].

2 The wave propagation problem on a plate

2.1 Formulation of the problem

In the ultrasonic wave propagation problem that we study, it is assumed symmetry with respect to the z-axis and zero displacement in the z-direction. Therefore, the wave propagation problem is represented as a two-dimensional plane strain model in the x−y plane [17, 18]. Physical damping is not included. The domain Ω of the problem is a finite 2D plate, as shown in the Figure 1, with boundary ∂Ω equal to the union of δ1, δ2, δ3 and δ4.

The pulse used to generate pure antisymmetric Lamb modes such as A0 is a signal in the y-direction at the left boundary δ4 (x = 0) of the plate. The pulse is described by the function g(t):

\[ g(t) = \phi \sin(2\pi f_0 t) \exp \left( -\alpha \frac{(t - T_0)^2}{T^2} \right) \]  

(1)

The parameters \( f_0, \phi \) defining \( g(t) \) are the frequency and the amplitude of the pulse respectively. Moreover, \( T_0 \) and \( \frac{T}{\sqrt{\alpha}} \) are the center and the width of the Gaussian factor of the pulse. Figure 2 shows a typical pulse \( g(t) \) for the frequency \( f_0 = 0.7 \text{ MHz} \). The parameters of the pulse \( g(t) \) depends on the frequency \( f_0 \) and are given in Table 1 and Table 2 of section 3.

The propagation of the pulse in the plate is modeled as a wave propagation problem. Hence, the displacement is a vectorial function \( \mathbf{u}(t, x, y) = (u_x(t, x, y), u_y(t, x, y)) \), depending on the

Figure 1: The physical domain Ω and its boundaries.

Figure 2: Pulse \( g(t) \) applied to the left boundary δ4 of the plate. The parameters of the pulse are \( f_0 = 0.7 \text{ MHz}, \alpha = 1.5, T_0 = 3.0 \cdot 10^{-6} \text{ s}, T = 2.0 \cdot 10^{-6} \text{ s} \) and \( \phi = 1.0 \cdot 10^{-3} \text{ m} \).
temporal variable $t$, the spatial variables $x = (x, y)$, the density of the material $\rho$ and the elasticity coefficients.

Transversally isotropic materials, such as carbon fiber reinforced polymer - one of the most important materials used in aerospace applications- depend on 4 elasticity coefficients, that in Voigt nomenclature are denoted as $C_{11}, C_{66}, C_{22}$ and $C_{12}$. Recall that if $C_{11} = C_{22}$ and $C_{11} = C_{12} + 2C_{66}$, then we recover the isotropic case.

In linear elasticity theory the strain tensor $S(u)$ is defined as,

$$ S(u) = \begin{pmatrix} S_{11}(u) & S_{12}(u) \\ S_{21}(u) & S_{22}(u) \end{pmatrix} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} \\ \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{pmatrix} \left( \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \right) $$

Moreover, according to Hooke’s law [17], the stress tensor $\sigma(u)$ can be written in terms of the strain as,

$$ \sigma(u) = \begin{pmatrix} C_{12}S_{22}(u) + C_{11}S_{11}(u) & 2C_{66}S_{12}(u) \\ 2C_{66}S_{12}(u) & C_{12}S_{11}(u) + C_{22}S_{22}(u) \end{pmatrix} $$

With this expression for $\sigma(u)$, the function $u(t, x, y)$ is solution of the system of partial differential equations of motion [17],

$$ \rho \frac{\partial^2 u}{\partial t^2} = \nabla \cdot \sigma(u) $$

In $\delta 4$ the Dirichlet boundary condition,

$$ u(t, x) = (0, g(t)), \quad x \in \delta 4 $$

is imposed. Moreover, we assume that boundaries $\delta 1, \delta 2$ and $\delta 3$ are free, which means that Neumann boundary conditions

$$ \sigma(u(t, x)) \cdot n(x) = 0, \quad x \in \delta 1, \delta 2, \delta 3 $$

hold, where $0 = (0, 0)$ and $n(x)$ is the vector normal to the boundary of the plate in the point $x$. Finally, the problem is solved in the temporal interval $t \in [0, t_f]$, with $t_f > 0$ given. The initial conditions

$$ u(0, x) = 0, \quad x \in \Omega $$

$$ \frac{\partial u}{\partial t}(0, x) = 0, \quad x \in \Omega $$

are also imposed.

### 2.2 Numerical solution of differential equations

In the literature, the most common approach to solve numerically transient problems consists in using a FEM discretization of the spatial variables, keeping continuous the temporal variable. That leads to a system of linear ordinary differential equations, which may be integrated by means of different methods. In the present work, we discretize first the second derivative of the displacement with respect to time $t$, building a uniform mesh in the interval $[0, t_f]$, with step $\Delta t$, and using a backward difference formula. For each fixed time $t = t_i$, we obtain a partial
differential equation in the spatial variables, which is solved with quadratic Lagrange FEM. The values of the time step $\Delta t$ and the mesh size $h$ used to solve the wave equations satisfy the bounds $h < \frac{\Lambda(f_0)}{20}$ and $\Delta t < \frac{1}{20f_0}$. The strategy leads to a sequence of linear systems with the same sparse, symmetric and positive definite matrix and right hand side depending on time $t_i$, see [19] for more details. In Figure 3 we show the deformation of a plate of carbon fiber reinforced polymer for $t = 7.3 \cdot 10^{-6}$ computed with our FreeFemm++ code available at [16].

Figure 3: Graphics for $t = 7.3 \cdot 10^{-6}$ of the deformation a carbon fiber reinforced polymer plate after emitting a pulse on the boundary $\delta A$. Colors correspond to the intensity of the deformation (the norm of the vertical displacement).

### 3 Phase velocity dispersion curve

In dispersive media the phase velocity of the wave depends on the frequency $f_0$. This dependence is described by the phase velocity dispersion curves. In the case of thin plates with constant thickness, the phase velocity dispersion curve $d(f_0)$ is the parametric curve,

$$d(f_0) = \left( \frac{L_y}{\Lambda(f_0)}, \frac{C(f_0)}{c_0} \right)$$

where $\rho$ is the mass density, $E$ Youngs modulus, $c_0 = \sqrt{\frac{E}{\rho}}$, $L_y$ is the thickness of the plate, $\Lambda(f_0)$ is the wavelength and $C(f_0)$ is the phase velocity, both depending on the frequency $f_0$.

According to international standards, dispersion curves must be available for NDT. In this sense, the FEM-PVM presented in this paper is very important for NDT applications dealing with homogeneous and isotropic plates, such as road pavements and plane metallic structures in nuclear plants. In these applications, dispersion curves computed with FEM-PVM may be used to evaluate flaws in thin plates, with a procedure based on a pitch-catch configuration [11]. The idea of this procedure is that in presence of flaws, such as delaminations, breaks in the material homogeneity produce, for a given frequency, a value of the measured phase velocity that does not agree with the value predicted by the dispersion curve.

In this section the FEM-PVM approach is used to compute $C(f_0)$ for a set of frequencies $f_0$. The general idea of this method is the following. For each value of $f_0$, the wave propagation equations for a pulse depending on $f_0$ are solved with FEM. From the FEM solution we obtain the displacements $u_y(t, p_i)$ of the plate in the vertical direction at selected points $p_i$ (receivers) on the top of the plate, as functions on time. The displacement functions are used to compute the arrival time at receivers of a fixed phase point of the pulse. Finally, with these arrival times and the distances between the receivers, the phase velocity $C(f_0)$ corresponding to the given frequency $f_0$ is obtained.
In the following two sections we show that FEM-PVM is useful to compute approximately the phase velocity dispersion curve for non isotropic plates and also for isotropic plates with more complicated geometry.

### 3.1 FEM-PVM for transversally isotropic plates with constant thickness

In this section we compute the phase velocity dispersion curve for a transversally isotropic plate of carbon fiber reinforced polymer with dimensions \( L_x = 5.0 \cdot 10^{-2} \) m and \( L_y = 1.0 \cdot 10^{-3} \) m and density \( \rho = 1250 Kg/m^3 \). The elastic coefficients for this material are \( C_{11} = 3.84 \cdot 10^{10} N/m^2 \), \( C_{12} = 3.0 \cdot 10^9 N/m^2 \), \( C_{22} = 7.7 \cdot 10^9 N/m^2 \) and \( C_{66} = 2.6 \cdot 10^9 N/m^2 \). To apply FEM-PVM, 4 receivers with coordinates \( p_i = (x_i, L_y) \), \( i = 1, \ldots, 4 \) with \( x_1 = 0.010 \), \( x_2 = 0.013 \), \( x_3 = 0.016 \) and \( x_4 = 0.019 \) are fixed on the top of the plate, see Figure 4.

![Figure 4: Rectangular plate with selected receivers \( p_i \), \( i = 1, \ldots, 4 \) on the top.](image)

The wave propagation equations are solved with quadratic FEM, for pulses \( g(t) \) with parameters given in Table 1, corresponding to 9 different frequencies \( f_0 \).

**Table 1: Parameters of the pulse applied at the boundary \( \delta \) on the plate.**

<table>
<thead>
<tr>
<th>( f_0 ) (MHz)</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.6</th>
<th>0.7</th>
<th>0.9</th>
<th>1.1</th>
<th>2.0</th>
<th>3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha ) (m)</td>
<td>2.5 \cdot 10^{-3}</td>
<td>1.5 \cdot 10^{-4}</td>
<td>8 \cdot 10^{-5}</td>
<td>1.1</td>
<td>1.5</td>
<td>2</td>
<td>3.2</td>
<td>10</td>
<td>45</td>
</tr>
<tr>
<td>( \phi ) (s)</td>
<td>10^{-6}</td>
<td>10^{-6}</td>
<td>10^{-6}</td>
<td>10^{-6}</td>
<td>10^{-6}</td>
<td>10^{-6}</td>
<td>10^{-6}</td>
<td>10^{-6}</td>
<td>10^{-6}</td>
</tr>
<tr>
<td>( T ) (s)</td>
<td>2 \cdot 10^{-6}</td>
<td>2 \cdot 10^{-6}</td>
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</table>

The results shown in Figure 5 correspond to \( f_0 = 0.7 \) MHz. Figure 5 left shows, for each fixed receiver \( p_i \), \( i = 1, \ldots, 4 \), the function interpolating the sequence of vertical displacements \( u_y(t_j, p_i) \) obtained for the values of \( t_j = j \Delta t \), \( j \geq 0 \). The arrival time \( \tilde{t}_i \) of the pulse at the point \( p_i \), \( i = 1, \ldots, 4 \) is computed as the value of \( t \) for a selected maximum (represented in Figure 5 as a bullet) of the function \( u_y(t, p_i) \). It holds that for a fixed value of \( f_0 \), the points \( (\tilde{t}_i, x_i) \), \( i = 1, \ldots, 4 \) are approximately on a line, the slope of this line is the phase velocity \( C(f_0) \) corresponding to the frequency \( f_0 \). Figure 5 right shows the points \( (\tilde{t}_i, x_i) \), \( i = 1, \ldots, 4 \) and the fitting line.

Repeating the previous methodology for the rest of the frequencies \( f_0 \) in Table 1, we compute the phase velocity \( C(f_0) \) corresponding to each frequency \( f_0 \). In Figure 6 we show with red circles the points \( d(f_0) \) obtained in (8) for the the numerical values of \( C(f_0) \) computed with FEM-PVM.

Observe that for the selected frequencies \( f_0 \) in Table 1, the points \( d(f_0) \) are in the region of high variation of the phase velocity and also in the region approaching the asymptote. The log–log transformation of the points \( d(f_0) \) computed from the numerical solution was fitted with a quadratic polynomial curve. The resulting fitting curve in the original variables is depicted in Figure 6.
Figure 5: Left: Graphic of the displacement in the vertical direction $u_y(t, p_i)$ of the selected receivers $p_i = (x_i, L_y)$, $i = 1, ..., 4$, for $f_0 = 0.7 \text{MHz}$. Red bullets: point on $u_y(t, p_i)$ corresponding to the arrival time $\tilde{t}_i$ of the pulse at the point $p_i$, $i = 1, ..., 4$. Right: Plot of $x_i$ versus arrival times $\tilde{t}_i$, for $i = 1, ..., 4$ and the fitting line.

Figure 6: Points $d(f_0)$ (circles) computed with FEM-PVM for all frequency values $f_0$ from Table 1. Dispersion curve (continuous line) obtained fitting the points $d(f_0)$.

3.2 FEM-PVM for isotropic plates with slowly varying thickness

As we already mentioned, the strategy explained at the beginning of section 3 is useful for plates with more general geometries. In this section we use FEM-PVM to compute approximately the phase velocity dispersion curve for an isotropic wedge. The geometry of the wedge is shown in Figure 7 and depends on the angle $\alpha$ that defines the slope of the wedge.

Figure 7: Wedge with slope depending on angle $\alpha$ and receivers $p_i$, $i = 1, ..., 5$ on the top.

We compute the dispersion curves of 4 aluminium plates ($\rho = 2700 \text{Kg/m}^3$) with small inclination angle $\alpha = 90^\circ$, 90.29°, 90.57°, 91.15° in one side, see Figure (7). The sizes of all these plates are $L_x = 0.2 \text{ m}$, $L_y = 2 \cdot 10^{-3} \text{ m}$ and $L_y = L_y + L_x \cdot \tan(\alpha - 90^\circ)$.

The wave propagation equations (4) are solved with quadratic FEM, for the pulses $g(t)$ obtained
The combination of Phase Velocity and Finite Element methods has proved to be successful to compute phase velocity dispersion curves for thin plates. We have shown that this approach is suitable for non isotropic rectangular plates and also for isotropic plates with slowly varying thickness. With the open source software FreeFem++ we have implemented our in-house code for the FEM-PVM approach, using quadratic Lagrange triangular finite elements.

Numerical results show for constant thickness plates that FEM-PVM provides accurate estima-
tions of phase velocity, in intervals with low frequencies and high variation of phase velocity, as well as in intervals of high frequency, where the phase velocity has an asymptotic behavior. In the case of wedges with small inclination angles $\alpha$ the numerical results confirm that the “adiabatic condition” is satisfied and that when the waves propagate in direction to decreasing thickness of the plate, the phase velocity values of the antisymmetric mode $A_0$ increase for increasing slope $\alpha$.

References


