A Hybrid Formulation Using The Transition Matrix Method and Finite Elements in a 2D Eddy Current Interaction Problem

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Abstract. A hybrid formulation of the eddy current interaction problem using the transition matrix method coupled to a finite element formulation of the defect is presented in a 2D geometry. The applied analytical method involves use of the free space Green's function to generate a system of boundary integral relations. The defect is enclosed in a circular region in the conductive material and this region is resolved and formulated using linear finite elements. The approach allows a complex defect to be described with a finite element contribution to the transition matrix representing the defect in the analytical analysis of the problem. The result combines the fast analytical computation with the possibility to describe a complex shaped internal defect with finite elements. The total transition matrix with the finite element contribution is calculated once initially and afterwards used in the analytical approach for calculation of the impedance response at various positions.

1. Introduction

The more efficient methods of computation for non destructive testing scattering problems such as analytical methods or the boundary element method (BEM) have restrictions on the geometry. That problem does not exist for the finite element method (FEM), which on the other hand is not as efficient. To get an efficient method that still is able to deal with complex defects or probes, hybrid methods are often used. A hybrid method where a finite element formulation of the defect is coupled to an eigenfunction formulation is described in [1]. In [2] FEM is used to get the incoming field from a complex probe and a boundary integral method is used to solve the problem. There also exist several papers that present BEM-FEM hybrid methods, see e.g [3], [4] and [5]. In the present paper a FEM – transition (T) matrix hybrid method is presented.

The T matrix gives the scattered field from the incoming field. When a defect is defined by its T matrix, very low cost simulations of the eddy current interaction can be done. The tricky part is most often to calculate the T matrix, especially for a defect of complex shape. The T matrix for a cylindrical or a spherical defect is straightforward to calculate analytically but other shapes demand a numerical calculation. To manage complex defects and still benefit from the efficiency of the T matrix method, FEM is here used to calculate the T matrix. In this way there are no longer any limitations in defect shape.

2. The T Matrix

The scattering geometry described in figure 1 is essentially the one considered in [6] extended with an inner scatterer \( S_3 \) inside the cylinder. The surface \( L_0 \) of the conducting material is
assumed to be a plane. The cylinder bounded by $L_1$ is now a mathematical surface only and does therefore not act as a scatterer. $L_1$ is just the boundary between the inner problem which is solved with the finite element method and the outer problem which is solved using the $T$ matrix formalism. The methods are connected by the surface field on $L_1$. The source is a rectangular conductor with currents in the $z$-direction.

To deal with the inner scatterer some modifications of the derivations in [6] need to be done, but the main part will remain. The notation $\psi = A_z$ is introduced for the $z$ component of the magnetic vector potential $A$. Then $\psi$ will satisfy Helmholtz equation (the time factor $e^{-i\omega t}$ is assumed throughout this paper)

$$\Delta \psi(r) + k^2 \psi(r) = -f,$$

where $k^2 = i\mu\sigma + \omega^2\mu \varepsilon$ and $f$ is the known source term. The subregions $S_i$, $i = 0, 1, 2, 3$ are assumed to be homogeneous, isotropic and linear. Then the magnetic permeability $\mu$, the electric permittivity $\varepsilon$ and the conductivity $\sigma$ are constants. Now integration over $S_0$ and $S_1$ and Green’s theorem yield the same integral representations as in [6]

$$-\int_{L_0} \left( G(r, r', k_0) \frac{\partial \psi^+_0(r)}{\partial n_0} - \psi^+_0(r) \frac{\partial G(r, r', k_0)}{\partial n_0} \right) \, dL_0 + \psi^{inc}(r') = \begin{cases} \psi_0(r'), & \text{if } r' \in S_0, \\ 0, & \text{if } r' \notin S_0, \end{cases}$$

$$-\int_{L_0} \left( G(r, r', k_1) \frac{\partial \psi^-_0(r)}{\partial n_0} - \psi^-_0(r) \frac{\partial G(r, r', k_1)}{\partial n_0} \right) \, dL_0$$

$$-\int_{L_1} \left( G(r, r', k_1) \frac{\partial \psi^+_1(r)}{\partial n_1} - \psi^+_1(r) \frac{\partial G(r, r', k_1)}{\partial n_1} \right) \, dL_1 = \begin{cases} \psi_1(r'), & \text{if } r' \in S_1, \\ 0, & \text{if } r' \notin S_1. \end{cases}$$

Here $\psi^\pm$ is the surface field on the positive ($+$) or negative ($-$) side of the boundary with respect to its normal vector $\hat{n}$. $G(r, r', k)$ is the free space Green’s function which satisfies Helmholtz equation (1) with a point source $f = \delta(|r - r'|)$.

To include the inner problem in the $T$ matrix solution would require one additional integration over $S_2$. This would restrict the defect to be of simple shape. Instead the inner problem will be solved with the finite element method. Continuing with the integral representations, all the involved functions are expanded in suitable basis functions. Here two sets
of basis functions are used. The plane wave basis functions which are defined as
\[ \Phi(k, r) = \frac{1}{\sqrt{8\pi}} e^{i k r}, \]
where the vector \( k \) is defined as \( (q, h) = k(\cos(\alpha), \sin(\alpha)) \) and \( r = r(\cos(\phi), \sin(\phi)) \).

The second set are the cylindrical wave basis functions
\[ \begin{align*}
Re \chi_n(r, k) &= \frac{\sqrt{\epsilon_m}}{2} J_m(kr) \begin{cases} 
\cos(m\phi), & \text{if } \zeta = e, \\
\sin(m\phi), & \text{if } \zeta = o,
\end{cases} \\
\chi_n(r, k) &= \frac{\sqrt{\epsilon_m}}{2} H_m^{(1)}(kr) \begin{cases} 
\cos(m\phi), & \text{if } \zeta = e, \\
\sin(m\phi), & \text{if } \zeta = o,
\end{cases}
\end{align*} \]
where \( \epsilon_n = 2 - \delta_{n,0}, J_m \) is a Bessel function and \( H_m^{(1)} \) is a Hankel function of the first kind. Here the multi-index \( n = m \zeta \) where \( m \in \mathbb{N}, \zeta = o \) denotes odd and \( \zeta = e \) denotes even, has been introduced.

The surface field on \( L_0 \) is expanded in the same way as in [6]
\[ \begin{align*}
\psi_0^+(r) &= \int_{C_+} \beta(\alpha) \Phi(k_0, r) d\alpha, \\
\frac{\partial \psi_0^+(r)}{\partial n} &= \int_{C_+} \gamma(\alpha) \frac{\partial \Phi(k_0, r)}{\partial n} d\alpha,
\end{align*} \]
where \( \beta \) and \( \gamma \) are the unknown expansion coefficients. Here \( C_+ \) is a complex integration contour from \( \alpha = \pi/4 + i\infty \) to \( 3\pi/4 - i\infty \) subject to \( \tan(\Re(\alpha)) = \coth(\Im(\alpha)) \) and \( C_- \) is defined as \( \alpha + \pi \in C_- \) if \( \alpha \in C_+ \). Defining the contour in this way leads to a real \( q \) component of the vector \( k \). From the definition it also follows that \( \Im(\delta) \geq 0 \) on \( C_+ \) and \( \Im(\delta) \leq 0 \) on \( C_- \) and therefore \( e^{i \delta y} \) will be bounded if \( y \geq 0 \) when \( \alpha \in C_+ \) and if \( y \leq 0 \) when \( \alpha \in C_- \).

Below the source the incoming field \( \psi_{\text{inc}}(r') \), can be expanded in downward traveling plane waves as
\[ \psi_{\text{inc}}(r') = \int_{C_-} a_{\text{inc}}(\alpha) \Phi(k_0, r') d\alpha, \]
where \( a_{\text{inc}} \) are expansion coefficients which can be determined from the source. The field at the mathematical boundary \( L_1 \) is expanded in cylindrical waves
\[ \psi_2^- = \sum_n \zeta_n \Re \chi_n(r, k_2), \]
\[ \frac{\partial \psi_2^-}{\partial n} = \sum_n \kappa_n \frac{\partial \Re \chi_n(r, k_2)}{\partial n}, \]
where \( \zeta_n \) and \( \kappa_n \) are the unknown expansion coefficients. In [6] the scatterer inside \( L_1 \) was not present and the derivative of the surface field could be calculated directly from the surface field. Now the derivative has to be expanded independent of the surface field with another set of expansion coefficients, \( \kappa_n \).

The formulation above is the same as in [6] except from expansion (11). \( \kappa_n \) can not be determined by differentiation of the surface field on the mathematical boundary as mentioned above, but the derivative of the field on \( L_1 \) can be determined from this surface field together with information of the inner scatterer. With the solution to this inner problem \( \kappa_n \) can be expressed in terms of \( \zeta_n \) as \( \kappa_n = \sum F_{mn} \zeta_m \), where \( F \) is a transformation matrix. If this is introduced in expansion (11) the outer problem can be solved in the same way as in [6]. It should be noted that the transformation matrix \( F \) is independent of the incoming field.
3. Finite Element Solution of the Inner Problem

The 2D finite element model is implemented using first order nodal elements. The solution of \( \psi(\mathbf{r}) \) is restricted to the limited domain \( S = S_2 \cup S_3 \) in space which is enclosing the defect. The right hand side of equation (1) is zero as no external currents are applied within \( S \). The goal is to calculate the matrix that gives the derivative \( \partial \psi(\mathbf{r}) / \partial \hat{n} \) at the boundary \( L_1 \) of \( S \) from the values of \( \psi(\mathbf{r}) \) along the boundary. Multiplying equation (1) with a test function \( w_i \) and integrating over the total surface \( S \) gives

\[
\int_{S} \left( \nabla w_i \cdot \nabla \psi(\mathbf{r}) - k^2 w_i \psi(\mathbf{r}) \right) dS - \int_{L_1} w_i \nabla \psi(\mathbf{r}) dL = 0. \tag{12}
\]

Expanding the magnetic vector potential in a set of linear basis functions gives

\[
\psi(\mathbf{r}) = \sum_{j=1}^{K} A_j \varphi_j(\mathbf{r}), \tag{13}
\]

where \( K \) is the total number of basis functions used in the expansion and \( j = 1, \ldots, N \) is the basis functions coupled to the external nodes according to figure 2. The numbering of the nodes and elements close to the boundary of the finite element problem is shown in the figure.

![Figure 2](image_url)

\textbf{Figure 2.} Numbering of nodes and external elements at the boundary between the inner formulation, solved by the FEM, and the analytical part of the eddy current scattering problem.

Usage of Galerkin’s method, i.e. \( w_i(\mathbf{r}) = \varphi_i(\mathbf{r}) \) for all nodes where \( \psi(\mathbf{r}) \) is unknown and substituting (13) into (12) gives a set of linear equations which can be represented in matrix form as \( \sum_j M_{ij} A_j = 0 \). The coefficients of \( M_{ij} \) are given by

\[
M_{ij} = \int_{S} \left( \nabla \varphi_i \cdot \nabla \varphi_j - k^2 \varphi_i \varphi_j \right) dS. \tag{14}
\]

Assume for simplicity that the number \( M = N \) in figure 2, so that no internal nodes are shared between several external elements.

In the finite element solution the magnetic potential in element \( i \) is defined by a plane with the normal \((\xi_i, \zeta_i, \eta_i)\). This normal for an element at the boundary is determined from the coefficients \( A_i, A_{i+1}, A_{N+i} \) in the three nodes of the element as

\[
\begin{align*}
\xi_i x_i + \zeta_i y_i + \eta_i A_i + c &= 0, \\
\xi_i x_{i+1} + \zeta_i y_{i+1} + \eta_i A_{i+1} + c &= 0, \\
\xi_i x_{N+i} + \zeta_i y_{N+i} + \eta_i A_{N+i} + c &= 0,
\end{align*} \tag{15}
\]
where \( c \) is a constant and \((x_i, y_i)\) are the coordinates of node \( i \). The coefficients \( A_{N+i} \) is unknown here but given by the finite element solution \( MA = 0 \) as

\[
\begin{bmatrix}
  M_{\text{ext}} & M_{\text{int}}
\end{bmatrix}
\begin{bmatrix}
  A_{\text{ext}}
  A_{\text{int}}
\end{bmatrix} = 0 \quad \Rightarrow \quad A_{\text{int}} = -M_{\text{int}}^{-1}M_{\text{ext}}A_{\text{ext}} = UA_{\text{ext}},
\]

where \( A_{\text{int}} \) is \([K - N \times 1]\) and \( A_{\text{ext}} \) is a \([N \times 1]\) vector. Here we have denoted the equation with parts that is relevant for the solution at the external and internal nodes with \( \text{int} \) and \( \text{ext} \), respectively. The solution of the coefficients for the internal nodes of the boundary elements \( E_1, \ldots, E_N \) in equation (15) is then

\[
A_{N+i} = \sum_{j=1}^{N} U_{ij}A_j.
\]

The coefficients \((\xi_i, \zeta_i, \eta_i)\) in equation (15) can now finally be calculated from a summation over coefficients of the magnetic potential in the external nodes represented in matrix form as

\[
\begin{align*}
\xi &= M_{\xi}A_{\text{ext}}, \\
\zeta &= M_{\zeta}A_{\text{ext}}, \\
\eta &= M_{\eta}A_{\text{ext}}.
\end{align*}
\]

The matrix coefficients are given by

\[
\begin{align*}
M_{\xi ij} &= y_i(\delta_{(i+1)j} - U_{ij}) + y_{i+1}(U_{ij} - \delta_{ij}) + y_{N+i}(\delta_{ij} - \delta_{(i+1)j}), \\
M_{\zeta ij} &= \delta_{ij}(x_{i+1} - x_{N+i}) + \delta_{(i+1)j}(x_{N+i} - x_i) + U_{ij}(x_i - x_{i+1}), \\
M_{\eta ij} &= x_i(y_{i+1} - y_{N+i})\delta_{ij} + x_{i+1}(y_{N+i} - y_i)\delta_{ij} + x_{N+i}(y_i - y_{i+1})\delta_{ij},
\end{align*}
\]

where \( i, j = 1, 2, \ldots, N \). The * notation indicates that \( \delta_{(N+1)1} = 1, x_{N+1} = x_1 \) and \( y_{N+1} = y_1 \) due to the closed circular boundary.

The derivative of the potential at the edges of the outer boundary is finally described by

\[
\frac{\partial \psi(r)}{\partial \hat{n}} = -\frac{\xi}{\eta}\hat{n}_x - \frac{\zeta}{\eta}\hat{n}_y = \left[ -M_{\eta}^{-1}M_{\xi}\hat{n}_x - M_{\xi}^{-1}M_{\eta}\hat{n}_y \right] A = T_{FEM}A_{\text{ext}}.
\]

The derived derivative at the outer boundary is constant along the boundary of each external element due to the selection of the linear basis functions. The derivative should be represented at the circular boundary used in the analytical model but is here approximated by the property of the nearby element. This will introduce an error which will decrease with increasing number of elements along the boundary. A defect is typically described by a specific distribution of the electrical conductivity within \( S \). The matrix \( T_{FEM} \) will represent the individual defect as described with a specific mesh. The matrix needs only to be calculated once. However, it should be noted that the matrix will be representative for the specific frequency used in equation (1) but will be independent of the position and orientation relative the conductor applying the source currents.

4. The Coupling at the Mathematical Boundary

To get \( F \) from \( T_{FEM} \) the fields expanded in the linear basis and in the cylindrical basis need to be coupled on the mathematical boundary. The first trivial step is to express the field on the external nodes \( A_{\text{ext}} \) in terms of the expansion coefficients \( \zeta_n \)

\[
A_{\text{ext}}(r_i) = \sum_n \text{Re} \chi_n(r_i, k_2)\zeta_n,
\]

5
Then as described above the derivative is found by applying the transformation matrix $T_{FEM}$

$$\frac{\partial \psi_j^-(r_j)}{\partial n} = \sum_{ni} T_{ji}^{FEM} \Re \chi_n(r_i, k_2) \zeta_n.$$  \hfill (22)

Now this derivative is projected on $\sin(m\phi)$ and $\cos(m\phi)$ to yield

$$\kappa_n = \sum_j \frac{2}{\sqrt{\epsilon_m H_m^{(1)}(kr)}} \int_{\phi_j}^{\phi_{j+1}} d\phi \frac{\partial \psi_j^-(r_j)}{\partial n} \begin{cases} \cos(m\phi), & \text{if } \zeta = e, \\ \sin(m\phi), & \text{if } \zeta = o. \end{cases}$$  \hfill (23)

Here the integration can be carried out analytically because of the piecewise constant approximation of the derivative $\frac{\partial \psi_j^-(r_j)}{\partial n}$. Thus the linear operator $F$ formally can be expressed as

$$F_{nm} = \sum_{ij} \int_{\phi_j}^{\phi_{j+1}} d\phi \frac{2 T_{ji}^{FEM} \Re \chi_n(r_i, k_2)}{\sqrt{\epsilon_m H_m^{(1)}(kr)}} \begin{cases} \cos(m\phi), & \text{if } \zeta = e, \\ \sin(m\phi), & \text{if } \zeta = o. \end{cases}$$  \hfill (24)

5. Numerical Calculations

In this section the solution of the field at the mathematical boundary $L_1$ is used to calculate the change of impedance due to a defect. This difference in impedance $\Delta Z$ can be calculated by integration over $L_1$ (see Auld and Moulder [7])

$$\Delta Z = \frac{1}{I^2} \int_{L_1} (E_{inc} \times H - E \times H_{inc}) \, dL,$$  \hfill (25)

On this surface the surface field expansion (10) can be used to reduce expression (25) to

$$\Delta Z = -\frac{i \omega}{\mu I^2} \sum_{mn} \zeta_n^{inc} \zeta_m \Re Q_{nm},$$  \hfill (26)

where

$$\Re Q_{nm} = \int_{L_1} \left( \frac{\partial \Re \chi_n(r, k_1)}{\partial n} \Re \chi_m(r, k_2) - \Re \chi_n(r, k_1) b_{12} \sum_l \frac{\partial \Re \chi_l(r, k_2)}{\partial n} F_{lm} \right) \, dL_1,$$  \hfill (27)

and $\zeta_n^{inc}$ is the expansion coefficients to the cylindrical wave expansion of the incoming field at the mathematical boundary. The important variable for the convergence of the calculated change in impedance $\Delta Z$ is the accuracy of the computed derivative at the boundary $L_1$ as a result of applying the $T_{FEM}$ matrix. This property depends on the element size along $L_1$ as well as the characteristics of the derivative itself. The defect selected for the convergence test of the hybrid formulation suggested here is represented as a quarter of a circle. Six different finite element meshes (A-F) representing the defect are evaluated. The number of boundary nodes is doubled between each mesh going from 24 to 48, 92, 180, 360 and finally 720. The configurations of mesh A, C and E are presented in figure 3. The $T_{FEM}$ matrix is calculated for the different mesh configurations.

The defect is represented as a region of zero electrical conductivity and the surrounding part, i.e. domain $S_2$, is given the conductivity of the bulk material which is here assumed to be 0.58 MS/m. The conductor applying the external source current in $S_0$ is rectangular with a width of 0.25 mm and a height of 1 mm. The conductor is positioned 0.1 mm above the bulk material. The circular region bounded by $L_1$ has a radius of 0.5 mm and is placed
with the centre at a depth of 0.6 mm. The defect is represented as a quarter of the circle placed with the center at a depth of 0.6 mm and with a radius of 0.4 mm. The problem is presented schematically in figure 1 and details of the three defect meshes are shown in figure 3. The resulting impedance change due to the defect is calculated for various positions of the conductor, carrying the source current, relative the defect. The result is shown in figure 4. The results based on the calculated $T^{FEM}$ matrix using mesh A, C and E is included in the figure and compared to the converged full finite element solution (denoted FEM).

**Figure 3.** Finite element mesh applied around the defect in the hybrid model.

**Figure 4.** Impedance as a function of the relative position between defect and source.

The relative error is presented in figure 5 in order to study how the solution converges as the number of boundary nodes is increased. The relative error is slightly above 10% for mesh A but shows a good agreement of the signal shape as the conductor is moved over the
defect. Improving the mesh is clearly giving an improved calculation of the $T^{FEM}$ matrix, the order of convergence is close to 1 which is clear from figure 5.

![Figure 5. Relative error as a function of the number of boundary nodes $N$, the order of convergence is close to 1.](image)

6. Conclusions

A hybrid formulation using the transition matrix method and finite elements is presented. This method gives the possibility to include complex defects and still allowing the representation of the probe and main parts of the problem to be analytical. The formulation presented here is 2D and it remains as further work to implement this in the more general 3D case. A second limitation is that the volume enclosing the defect, represented with finite elements, cannot be surface breaking using the formulation as presented within this paper and remains as an extension of this approach. However, the potential of the mathematical method is that the parts of the solution that are time consuming to calculate will only have to be considered once. The impedance as a function of the parameters related to the analytical model such as the probe or defect position and orientation can be computed efficiently.

References