Investigation of the validity of the elastic Kirchhoff approximation for rough cracks using a finite element approach

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Abstract
The Kirchhoff approximation (KA) to calculate the elastic wave scattering from 2D and 3D rough cracks is examined by a comparison with a finite element (FE) approach. This approach couples a time domain finite element solver and a hybrid method to compute the scattering signals from rough cracks. Random rough surfaces with Gaussian profiles are used in this paper to study the validity of the Kirchhoff approximation. Simulations are run as a function of incident/scattering angle and roughness. Both the shape and the peak amplitude of the received signal are compared using the two different numerical approaches. Certain restricted ranges for the Kirchhoff approximation are found through the comparison with the FE method, and these are found to be different between 2D and 3D.

Keywords: Rough crack, Scattering, Kirchhoff approximation (KA), Finite element (FE)

1. Introduction
The Kirchhoff approximation (KA) is an analytical methodology to calculate the scattering behavior from an obstacle with an arbitrary shape. It has been widely used in many fields to produce fast prediction of the scattering waves from rough surfaces, such as underwater acoustics [1], remote sensing [2] and seismology [3]. Researchers developing modelling tools for NDE in the power industry in UK have observed significant differences of several dB between experimentally measured reflections from rough defects and simulations of the same cases using three-dimensional (3D) KA [5]. While this is properly expected to be due to the approximations of KA, it is important to understand the range of validity, and thus when the use of KA should be expected to be reliable. The motivation of this study is to investigate the fundamental aspects of the elastic KA by comparison with a reference numerical method. Some literature can be found regarding the evaluation of the accuracy of the KA [1, 4] and most of the works are limited in using 2D rough surfaces. However, the scattering behavior from a 3D rough crack is more realistic for inspections but more complicated as well. In this paper, we present a numerical study of the evaluation of the Kirchhoff model from 3D random rough surfaces and compare the results with the corresponding 2D cases.

We use the finite element (FE) method as the reference because it can represent the exact solution once the convergence criteria are satisfied. The main two challenges here are to represent the rough defect with a reasonable 3D mesh and to reduce the computation cost with a 3D FE solver. In this paper, a finite element boundary integration (FEBI) method is used to significantly reduce the memory requirement. Also, a time domain FE explicit solver is incorporated into multiple GPUs [6] to accelerate the computation speed. In addition, a mixed meshing algorithm is implemented to generate the 3D mesh with improved efficiency. With all these efforts, Monte Carlo simulations are realized for multiple rough defects and the computed results are compared between FEBI and KA.
2. Kirchhoff Theory

The fundamental assumption of the Kirchhoff theory is to approximate any surface point as an infinite long tangential plane based on the local gradient as shown in Fig. 1. Once the reflection coefficient at each point is known, the total displacement can be calculated as a summation of the incident wave and reflected P and S waves:

\[ u_t = u_0 (d_0 + A_{pp} d_p + A_{ps} d_s) \]  

where \( u_t \) represents the total displacement and \( u_0 \) is the amplitude of the incident P wave. \( A_{pp} \) and \( A_{ps} \) are reflection coefficients of P and S waves respectively. \( d_0, d_p \) and \( d_s \) are the displacement polarization vectors for the incident P and reflected P/S waves. By substituting Eq. (1) into the Helmholtz integration equation for elastic waves with a stress-free boundary condition, the scattering waves can be calculated using the following equation:

\[ u_{k}^{sc} = \int_{S} \Sigma_{ijk} (|R - r|) u_i(r) n_j(r) dS(r) \]  

where \( \Sigma_{ijk} \) is the stress Green’s tensor, \( u_i \) is the ith component of the total displacement, \( n_j \) is the jth component of the unit normal vector surface pointing towards the observation point at \( R \). After discretization with a spatial interval \( dx = \lambda/30 \) for 2D rough surfaces and \( dx = dy = \lambda/16 \) for 3D rough surfaces, the scattering displacement at \( R \) can be computed explicitly. In reality, Eq. (2) is implemented in the frequency domain and the results are synthesized back to the time domain to obtain the scattering signals using the inverse fast Fourier Transform (IFFT).

3. Finite Element Model

The finite element method is a very general numerical tool which can cover most aspects of elastic wave problems, and it has been used in the work reported in this paper to evaluate the Kirchhoff theory as a reference. However, it is well known that a full 3D FE model is very computational expensive and even the 3D free meshing of the irregular crack is difficult. We therefore use a FEBI method implemented in a GPU driven software Pogo [6], with a mixed meshing algorithm in 3D to successfully overcome these difficulties. Fig. 2 (a) shows a sketch of the FEBI method to simulate a general ultrasonic inspection problem. The incident wave from the transducer is calculated by the Rayleigh integration. A source line in 2D or a source plane in 3D is placed just \( \lambda \) above the crack to excite the FE model and produces exactly the same incident wavefield from this transducer. Specific time traces of forces are calculated from a finite difference time domain method (FDTD) and are applied into each node at the source line. The time domain FE computation is executed by either Pogo or commercial FE software inside the small FE box. A set of monitoring nodes are located at the crack surface to record the time series of the boundary displacement, which are later substituted into the boundary integration formula Eq. (2) again to calculate the scattering signals. The details of
this method are not included here, for brevity, but have been submitted for publication elsewhere [7].

![Illustration of the FE method. a) General FEBI model, b) Modified model for the scattering from a rough backwall.](image)

The Kirchhoff model is known to be inaccurate in calculating the tip diffracted waves, which might be dominant especially at near grazing angles. In this paper, we are focusing on the evaluation of the reflection from the surface itself, thus the tip diffracted waves need to be eliminated. The FEBI method is used here but with some modifications to calculate the scattering from a rough backwall as shown in Fig. 2(b). One benefit of this model is that since the backwall is continuous, the physical tip diffraction phenomena can be significantly reduced. A rough backwall is surrounded by smooth surfaces extended to the absorbing region. A spatial Hanning window is multiplied to the whole bottom surface to make a smooth transition between the rough part and the smooth part. A source line is placed above the backwall to excite a compressional plane wave. It should be noted here that the two tips of this source line need to be buried into the absorbing region to prevent any unwanted circular waves generated from the two ends. This is crucial because the local FE region is very small thus any polluting waves would ruin the results. Before taking the boundary displacement into Eq. (2), the same Hanning window function is multiplied, resulting in a zero value of the displacement at the boundary of the rough surface. This can effectively mitigate the numerical ‘edge effects’ caused by integration along a contour with a finite length. The model shown in Fig. 2(b) is in 2D. It is extended to 3D by replacing the FE rectangular box with a cubic box and the 2D surface with a 3D surface.

4. Comparison between FE and Kirchhoff

The surfaces used in this study follow the classical Gaussian distribution, which can be generated by the RMS $\sigma$ and the correlation length $\lambda_0$. RMS $\sigma$ determines the height scale of the surface and the correlation length $\lambda_0$ is a measurement of how two surface points relate with each other within a certain distance. The moving average method [8] is used here to produce surfaces with a variety of roughness. In this paper, four different rough surface profiles are tested with the RMS $\sigma = \lambda/8, \lambda/6, \lambda/5$ and $\lambda/3$, while the correlation length $\lambda_0$ is fixed as $\lambda/2$. Numerous simulations are run with the incident angles $\theta_i$ ranging from $0^\circ$ to $30^\circ$ to calculate scattering signals with $\theta_s$ from $-90^\circ$ to $90^\circ$. For each roughness, 50 realizations of rough surfaces are generated and Monte Carlo simulations are performed to obtain a statistically meaningful result for comparison. Previous studies computed the coherent contribution [1, 2, 4] of the scattering signal by averaging both the amplitude and the phase of the signals from all realizations. In real NDT inspection scenarios, we are more focusing on the amplitude rather
than the phase because the amplitude directly determines the probability of detection. The statistical parameter used for comparison is therefore the ensemble averaging of the peak of the Hilbert transformed scattering signals.

4.1 2D Comparison Result

Simulations are first run using 2D surfaces and the corresponding FE model is meshed by linear triangular element (CPE3 in Abaqus (Dassault Systemes Simulia Corp., Providence, RI)) with an element size of $\lambda/30$. With a proper partition, only the region surrounding the rough backwall is free meshed and the remaining region can be regularly meshed. A five-cycle Hanning windowed compressional wave with the center frequency at 4MHz is used and the length of the 2D rough surface is about 5.2$\lambda$. Fig. 3 shows the comparison of the averaged amplitude with respect to the scattering angle at the normal incidence angle ($\theta_i =0^\circ$), with different profiles of roughness. The amplitude is in dB scale and the reference case is the normal pulse echo response from a smooth surface of the same size. As can be seen from Fig. 3(a), (b) and (c), there is a good agreement between the KA and the FE results except at near grazing angles. If the tolerance of error is set to be 1dB, the acceptable range of the scattering angle $\theta_s$ is from -70° to 70°. However, if the RMS $\sigma$ increases to $\lambda/3$, even at the specular direction the KA no longer matches with the FE. This suggests that for a normally incident wave, the KA is valid when $\sigma < \lambda/3$. In addition, at near specular directions the KA underestimates the amplitude but at near grazing angles it overestimates the amplitude compared with the FE results. This can be explained by the fact that the KA assumes the shadowing effects of scattering waves and neglects any multiple reflections. As a result, less energy is reflected back to the normal or near normal scattering angles and more energy is distributed at near grazing angles.

**FIGURE 3.** 2D Comparison of the averaged amplitude with respect to the scattering angle between the Kirchhoff theory and the FE method with a normal incidence angle $\theta_i = 0^\circ$. a) $\sigma = \lambda/8$, b) $\sigma = \lambda/6$, c) $\sigma = \lambda/5$ and d) $\sigma = \lambda/3$. 


Figure 4 shows the comparison results of the amplitude but with a slightly oblique incidence angle ($\theta_i = 30^\circ$). Excellent agreement can be found from all four cases even with a very rough surface as can be seen in Fig. 4(d), which is different from the case with the normal incidence angle as shown in Fig. 3(d). This finding indicates that even with a very rough crack, an acceptable inspection result may still be achieved using the Kirchhoff model with a small oblique incidence angle.

![Comparison Results](image)

**FIGURE 4.** 2D Comparison of the averaged amplitude with respect to the scattering angle between the Kirchhoff theory and the FE method with an oblique incidence angle $\theta_i = 30^\circ$. a) $\sigma = \lambda/8$, b) $\sigma = \lambda/6$, c) $\sigma = \lambda/5$ and d) $\sigma = \lambda/3$.

### 4.2 3D Comparison Results

The 2D simulation results shown above are equivalent to the 3D models using corrugated surfaces which have a height variation only in one direction. Such simplifications have been performed in the past to ease computational demands, but in reality all defects have roughness in two dimensions. It is therefore necessary to evaluate the performance of the Kirchhoff with real random rough defects and this requires corresponding 3D FE models. The FEBI model as discussed and implemented with a 2D surface can be extended to three dimensions with the rectangular FE region replaced by a cubic FE box and the source line replaced by a source plane. However, taking one more dimension into account leads to more complexity with the preprocessing of the FE model. Extra efforts must be taken to build an automatic preprocessing procedure including constructing 3D CAD geometry of the FE model and an efficient 3D meshing algorithm. The 3D CAD software Rhino (Robert McNeel & Associates, Seattle, WA) was implemented in this study to build the rough surfaces as shown in Fig. 5.
Free meshing algorithms using linear tetrahedron elements have been commonly utilized in previous studies for the 3D FE modelling of rough surfaces \[2, 9\]. As is well known the free meshing tends to be random and sometimes it will fail to mesh the whole region with a surface of high roughness. A new mixed meshing algorithm combining two meshing algorithms is therefore developed to generate a 3D mesh with improved efficiency. The region just above the 3D rough surface is meshed via a free meshing algorithm to produce elements around the surfaces. This very local meshing profile is then used as an input to a Matlab (MathWorks, Natick, MA) code to grow a regular mesh to fill up the remaining region of the 3D FE model. The regular meshed region is meshed with many hexahedron cells and each cell is composed of six linear tetrahedron elements as shown in Fig. 6(a). Fig. 6(b) shows a local view of the mesh profile of one 3D FE model. It should be noted that at the boundary of the free meshing and the regular meshing regions, the two neighboring elements need to have the same hypotenuse to make the mesh compatible. In this manner, the mesh minimizes the complexity caused by the free meshing algorithm but still captures the exact shape of the complex rough surface.

In a similar manner as was deployed for the 2D studies (corrugated surfaces), a Monte Carlo simulation is performed with various roughness and incidence/scattering angles, and the averaged amplitudes from all the realizations are compared between the KA and the FE methods. Fig. 7 shows the comparison for the case of a normally incident wave. Good agreement can be seen from -70° to 70° when \(\sigma = \lambda/8, \lambda/6\) and \(\lambda/5\). However, when the RMS value increases to \(\lambda/3\), the agreement no longer exists even at the normal backscattering direction. This indicates that for normal incidence, the 3D Kirchhoff approximation is valid when \(\sigma < \lambda/3\),
which is consistent with the criteria from 2D simulations as shown in Fig. 3. In addition, as can be seen from Fig. 3 and Fig. 7, the amplitude of scattering signals from 3D surfaces becomes several dB smaller than the corresponding 2D surfaces as the roughness increases. This finding indicates that the 3D rough surface attenuates more than the 2D surface. If the incidence angle is slightly oblique at 30°, it can be seen in Fig. 8 that the FE and the KA models show agreements at \( \sigma = \lambda/3 \), but only within a very narrow angular range around the specular direction, roughly from 30° to 50°. This is different from the corresponding 2D cases as shown in Fig. 4(d). In 2D situations, the acceptable region of the scattering angles is much larger ranging from -65° to 65°. This reduced angular range in 3D is caused by the fact that adding roughness in one more direction would lead to more multiple reflections and local resonances as well, which would reduce the accuracy of the 3D Kirchhoff theory. On the other hand, this again shows that a slightly oblique incident wave can improve the performance of the Kirchhoff model.

**FIGURE 7.** 3D Comparison of the averaged amplitude with respect to the scattering angle between the Kirchhoff theory and the FE method with a normal incidence angle \( \theta_i = 0° \). a) \( \sigma = \lambda/8 \), b) \( \sigma = \lambda/6 \), c) \( \sigma = \lambda/5 \) and d) \( \sigma = \lambda/3 \).
FIGURE 8.  3D Comparison of the averaged amplitude with respect to the scattering angle between the Kirchhoff theory and the FE method with an oblique incidence angle $\theta_i = 30^\circ$. a) $\sigma = \lambda/8$, b) $\sigma = \lambda/6$, c) $\sigma = \lambda/5$ and d) $\sigma = \lambda/3$.

5. Summary and Conclusion

In this paper, the elastic Kirchhoff theory is evaluated by comparison with a FE method using both 2D and 3D rough surfaces. A FEBI method is proposed and implemented in conjunction with a GPU driven time domain explicit solver to simulate the elastic wave scattering in 3D. Monte Carlo simulations are run with multiple roughness and incident/scattering angles and the averaged amplitudes of the scattering signals are used for comparison. Generally speaking, with a normally incident wave the Kirchhoff theory is valid when $\sigma < \lambda/3$ except the grazing angles in both 2D and 3D. Furthermore, with a small oblique incidence angle the 2D Kirchhoff theory is accurate when $\sigma = \lambda/3$ with an scattering angular range from $-65^\circ$ to $65^\circ$, while the 3D Kirchhoff theory is also valid when $\sigma = \lambda/3$ but only around specular scattering angles. In future work, surfaces with different correlation lengths will be tested for a more strict comparison. Also we are interested in using either the FE or the Kirchhoff model to study the statistical properties of the scattering signals from 2D and 3D rough surfaces with known height distributions.

References

5. EDF Energy (private communication).