Compressive sensing sparse sampling method based on principal component analysis

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Abstract

The rapid development of aerospace has increased the technical requirements for aerospace structures. Structural health monitoring technology, including monitoring, detection and identification of structures, has become a key technology to detect the structural safety performance of aerospace structures. Early detection techniques were limited by the structure and performance requirements of the examined materials. Therefore, the detection of complex components is very difficult. Ultrasonic phased array technology uses multi-element transducer with different shapes, can improve the shortage of the early detection technology and realize the detection of the inspected structure with high-speed, omnidirectional and multi angle. However, due to the omnidirectional and multi angle characteristics of ultrasonic phased array technology, the amount of information obtained by scanning will be enormous, and will increase the difficulty of information processing. As a result, effective processing method of large amounts of information is critical. Compressive sensing theory pointed out that only when the signal is sparse in the time domain or a transform domain, the signal can be sampled with the sampling rate much less than the sampling rate of traditional Nyquist sampling theorem, and reconstructed with high probability. That is, the sparsity of signal is the precondition of compressive sensing. Most of signal is not sparse in natural society, but can be expressed as sparse form by some kind of sparse transformation. Commonly used sparse transform, such as Discrete Fourier Transform, Discrete Cosine Transform, and so on, will lose some information after sparse transformation, because these transform bases used in the transformation are generally fixed. However, using principal component analysis method for data reduction, the new variable with low dimension which is linear correlation with the original variable is selected to instead of the original variable with high dimension for the data reduction. So that the useful data of the original signal can be included in the sparse signal after dimensionality reduction with maximize portability. Therefore, the signal after sparse representation can reduce the loss of data as much as possible, and is conducive to improve the efficiency of signal reconstruction. Finally, the composite material plate was used for the experimental verification. The experimental result shows that the sparse representation of signal based on principal component analysis can reduce signal distortion and improve signal reconstruction efficiency.

Keywords: Principal component analysis, Compressive sensing, Sparse representation, Signal reconstruction

1. Introduction

Principal component analysis (PCA) is one of the most commonly used multivariate statistical techniques [1]. In 1901, Pearson [2] proposed first in the study of space in a linear and plane of the best simulation. Then, improved by Hotelling [3], the dimensionality reduction and feature extraction of multivariate statistical methods was formed. Principal component analysis uses an orthogonal mathematical transformation to convert the observed values of a set of possible dependent variables to the values of the variables that are not linearly related which is called the principal component. The number of principal components is less than or equal to the number of original variables. Only when the data is combined with normal distribution, the principal component is independent of each other. PCA is sensitive to the correlation degree between the original variables, and is also called Hotelling transform, discrete KLT transform or proper orthogonal decomposition in different fields.
By projecting the data into the low dimensional space and the most possible features of the original data, PCA can be used to deal with the data of high dimension, noise and high correlation, and become the most widely used technology in the field of process monitoring. So far the development has become a kind of exploratory data analysis and prediction model of the effective tool by feature extraction using covariance or correlation matrix decomposition method, or the use of a set of data matrix signal value. In recent decades, scholars have made a deep research on the characteristics of PCA extraction and dimension reduction. It is the development of PCA theory and it is widely used in different disciplines [4, 5, 6]. Wold et.al [7] used the method of cross examination to determine the number of PCA principal component, and the method based on the PCA for model prediction. Ku [8] introduces the method of "time lag transfer" to the field of statistical monitoring, and extends the monitoring method of the previous static PCA to the dynamic PCA method, which is applied to the detection of the disturbance of the dynamic multivariable system.

Compressive sensing (CS) pointed out that as long as the signal is sparse, then through the sampling rate is far lower than that of the traditional Nyquist sampling theorem to collect signals, and then complete the reconstruction of signals by reconstruction algorithm [9,10,11]. The theory must be premised on the sparsity of the signal, the principal component analysis can be used for data dimensionality reduction, therefore, the compressed sensing method based on principal component analysis is proposed in this paper, the ultrasonic phased array structural health monitoring process, for sparse data representation problem gives a better solution.

2. Principal component analysis

Principal component analysis method is based on the original data space, to reduce the dimension of the original data space by constructing a new set of latent variables, and then from the mapping space sample new major changes in information extraction, statistical features, which constitute the original data of the spatial characteristics of understanding. The variable of the new mapping space is composed of the linear combination of the original data variables, which greatly reduces the dimension of the projection space. Because the statistical characteristic vectors of the projection space are orthogonal to each other, the correlation between variables is eliminated, and the complexity of the original process characteristic analysis is simplified.

The basic idea is to find a new set of variables instead of the original one, and the new variable is a linear combination of the original variables. From an optimization point of view, the number of new variables is less than the original variables, and to maximize the amount of useful information to carry the original variable, and the new variables are not related to each other. Its contents include the definition and acquisition of main elements, as well as through the principal component of the data reconstruction. This method can effectively identify the most important elements and structures in the data, remove the noise and redundancy, reduce the original complex data, and reveal the simple structure behind the complex data.

Assuming that \( X \) is a \( N \times M \)-dimension matrix, each column corresponds to a variable, and each row corresponds to a sample. Then the matrix \( X \) can be decomposed into the sum of the outer product of \( M \) vectors, and the outer product is the product of the two equal length vectors, where must make the column be multiplied by the row, as shown below.

\[
X = t_1 p_1^T + t_2 p_2^T + \cdots + t_m p_m^T
\]

where \( t_i \) is the score vector, \( p_i \) is the load vector, and the score vector of \( X \) is called the principal component of \( X \).

Each of the score vector is orthogonal, that is, for any \( i \) and \( j \), when \( i \neq j \), it can meet \( t_i^T t_j = 0 \).

Then each of the load vector is also orthogonal, and the length of each load vector is 1, as shown below.
Given the original data $x = (x_{ij})_{N \times M}$, making the standardization of $x$ to eliminate the dimensional effects, and the expression is shown as follow.

$$x'_j = \frac{x_{ij} - \bar{x}_j}{S_j}$$

where $\bar{x}_j = \frac{1}{n} \sum_{i=1}^{n} x_{ij}$, and $S_j^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_{ij} - \bar{x}_j)^2$, $j = 1, 2, \ldots, m$.

Making calculation of the correlation coefficient matrix between the data variables after standardized operation, that is, the covariance matrix $R$, is shown below.

$$R = \begin{bmatrix}
  r_{11} & r_{12} & \cdots & r_{1m} \\
  r_{21} & r_{22} & \cdots & r_{2m} \\
  \vdots & \vdots & \ddots & \vdots \\
  r_{m1} & r_{m2} & \cdots & r_{mm}
\end{bmatrix}$$

where the element $r_{jk}$ represents the correlation coefficient of the original variable $x'_j$ and $x'_k$, and $r_{kk} = 1$, as shown below.

$$r_{jk} = \frac{\sum_{i=1}^{n} (x'_{ij} - \bar{x}_j)(x'_{ik} - \bar{x}_k)}{\sqrt{\sum_{i=1}^{n} (x'_{ij} - \bar{x}_j)^2 \sum_{i=1}^{n} (x'_{ik} - \bar{x}_k)^2}}, \quad i, j = 1, 2, \ldots, m$$

The elements on the diagonal are corresponding to the variance of the observation variables, and the non diagonal elements are corresponding to the covariance between the observation variables. Assuming that the large variance of the diagonal is a signal, and the small variance is a noise, if the element on the diagonal is greater, the signal is stronger, and the importance of the variable is higher. Otherwise, if the element on the diagonal is smaller, it indicates that there may be noise or secondary variables. The element on the non diagonal corresponds to the redundancy degree between the correlated observation variables.

Jacobi method is used to solve the characteristic equation $|L - R| = 0$, the eigenvalues of the covariance matrix and the corresponding eigenvectors are obtained. Then make it according to the size of the order, the characteristic value is recorded as $\lambda_1, \lambda_2, \ldots, \lambda_m$, and the corresponding feature vector is recorded as $p_1, p_2, \ldots, p_m$.

Then calculate the main elements $t_i = Xp_i$, where the principal component $t_i$ on behalf of the projection of the data matrix $x$ on the direction of the load vector corresponding to the main element. If the length of projection is greater, the degree of coverage or the scope of the change is larger in the direction of $p_i$. And then, if $\|p_1\| > \|p_2\| > \cdots > \|p_m\|$, $p_1$ represents the maximum direction of the change of $x$, pm represents the smallest direction of the change of $x$. Then calculating the contribution rate of each principal components is $\frac{\lambda_i}{\sum_{k=1}^{m} \lambda_k}, i = 1, 2, \ldots, m$, as well as...
the cumulative contribution rate is \( \sum_{k=1}^{m} \frac{\lambda_k}{\lambda_1} \), \( i=1,2,\cdots,m \). In general, the 1-th, 2-th, \( k \)-th principal component corresponding to the eigenvalues of \( \lambda_1, \lambda_2, \cdots, \lambda_k \) will be selected, where the cumulative contribution rate of eigenvalues is between 85% and 95%.

In addition, according to the needs, the corresponding dimension (that is, the number of principal components) is selected to composition of the transformation matrix, as shown bellow.

\[
A = \begin{bmatrix}
    r_{11} & r_{12} & \cdots & r_{1k} \\
    r_{21} & r_{22} & \cdots & r_{2k} \\
    \vdots & \vdots & \ddots & \vdots \\
    r_{m1} & r_{m2} & \cdots & r_{mk}
\end{bmatrix}
\]  

(7)

Finally, the new data after dimension reduction is calculated as bellow.

\[
s = A^T \cdot x
\]  

(8)

3. Compressive sensing method based on principal component analysis

Compressive sensing is a novel theory of sampling and restoration for sparse signal [9,10,11]. As long as the original signal is sparsity in the time domain or under some kind of orthogonal transform, the signal can be sampled in low sampling rate, and the original signal can be reconstructed with high probability.

3.1 Sparse representation of signal

Compressive sensing theory is based on the premise that the signal must be sparse. When making sparse representation of the signal, the appropriate sparse transform base according to the signal characteristics is necessary to be selected. The principal component analysis method makes a kind of high dimensional data reduce for low dimensional data. Then, looking for a set of new variables in low dimensional to replace the original variables in high dimension, where the new variables must satisfy the conditions associated with the original variables. Therefor, the new variables can carry the maximum information of original variable. According to the principal component analysis, this method can be used to make sparse representation of the signal. Compared with the commonly used sparse representation method, the sparse signal obtained by the proposed method is more closely related to the original signal.

Suppose an original \( x \) signal with the length of \( N \), the number of signal is \( M \), a \( N \times M \) dimension matrix can be constructed with the original signal, because there are mutual relationship between the amplitude of each signal in each time point. According to the principal component analysis, the covariance matrix obtained of \( N \times N \) dimension can be used as the sparse transform based \( \Psi \) for sparse representation of signal. Then, the original signal \( x \) can be expressed as below.

\[
x = \sum_{i=1}^{N} \theta_i \psi_i \quad \text{or} \quad x = \Psi \Theta
\]  

(9)

where \( \Psi = [\psi_1, \psi_2, \ldots, \psi_N] \) is the transformation matrix of \( N \times N \) dimension, \( \Theta \) is the sparse coefficient vector obtained by \( x \) according to the principal component analysis, and must meet the following formula,

\[
\Theta = \Psi^T x \quad \text{or} \quad \theta_i = < x, \psi_i > = \psi^T_i x
\]  

(10)
3.2 Projection observation of the signal

The core of the compressive sensing theory is the design of the measurement matrix, and can directly determine whether the compressive sensing can be implemented successfully. If the signal $x$ has a sparse representation under an orthogonal transform $\Psi$, a measurement matrix $\Phi \in R^{M \times N}$, which is not related to the transform base $\Psi$ can be given, and a linear measurement of $M$ dimension can be obtained, as shown below.

$$y_i = < \theta, \phi_i >$$ (11)

Suppose the production measurement vector is $y = (y_1, y_2, \ldots, y_M)$, then

$$y = \Phi \Theta = \Phi \Psi^T x$$ (12)

Sufficient information about the signal $x$ is included in these limited observations, and if making $\Xi = \Phi \Psi^T$, the formula (12) can be converted as follow.

$$y = \Phi \Psi^T x = \Xi x$$ (13)

where $\Xi$ is called the sensing matrix.

In order to restore the original signal with high probability, the production measurement matrix $\Phi$, which is not related to the sparse transform based $\Psi$ and satisfied with the restricted isometry property, is needed to be constructed to make production transformation of the signal. Gauss random measurement matrix is not related to the majority of fixed orthogonal base and satisfies the restricted isometry property, so the Gauss matrix can be used as the projection observation matrix [12,13,14]. For the ultrasonic phased array signal, the Gauss random measurement matrix is multiplied with the sparse coefficient of the phased array signal, and the observation vector of the signal can be obtained.

Suppose the measurement matrix $\Phi$ is $M \times N$ dimension, and $\Phi \in R^{M \times N}$, then the general term displayed as below.

$$\Phi(i, j) = \frac{1}{\sqrt{M}} h_{ij}$$ (14)

Each element in the matrix is independent to the Gauss distribution with the mean value of 0, and the variance of $1/\sqrt{M}$. This matrix is not related to the vast majority of sparse signal, and requires less measurement values in the reconstruction. Gauss random measurement matrix is a matrix with very strong randomicity, but the disadvantage is high uncertainty. For a signal with a length of $N$ and a sparse degree of $K$, only $M \geq cK \log(N/K)$ measured values are needed to recover the original signal with high probability, where the $c$ is a very small constant.

3.3 Sparse reconstruction of the signal

During the process of compressive sensing, reconstructing the signal $x$ from the observations $y$ is the inverse problem relative to compression sampling, and is called the signal reconstruction. By solving the equation (13), the reconstructed signal can be obtained. This problem is underdetermined with infinite solutions. Candes et al. proved that the underdetermined problem can be solved by solving the minimum $l_0$-norm, that is shown below.

$$\min \| \Theta \|_0 \quad \text{s.t.} \quad y = \Phi \Theta = \Phi \Psi^T x$$ (15)

Formula (15) is a linear programming problem, and is also a convex optimization problem. Making the reconstruction error take into account, it is converted into a minimum $l_1$-norm problem, as displayed follow.

$$\min \| \Theta \|_1 \quad \text{s.t.} \quad \| \Phi \Theta - y \|_2 \leq \epsilon$$ (16)

During the process of signal reconstruction, convex optimization algorithm and greedy iterative algorithm are commonly used [15,16,17]. One kind of algorithm is based on convex optimization, mainly by increasing the constraint to obtain the sparsest. And commonly used algorithms are basis pursuit algorithm and gradient projection sparse reconstruction algorithm. The other kind of algorithm is based on greedy iterative algorithm, mainly by the combination
of local optimization method to find the non-zero coefficients, in order to approach the original signal. Commonly used algorithms are matching pursuit algorithm and orthogonal matching pursuit algorithm.

4. Experiment and results

In this session, the composite plate works as the experimental object. There are nine piezoelectric elements in the linear array arranged on the plate with 12 mm of adjacent equal interval. In signal acquisition, data collection points are 1024, and sampling frequency is $f_s=1000000\text{Hz}$.

Making one array element as a drive to transmit signal, and the other eight elements as the sensor to receive the reflection signal. Each array element stimulates the signal in turns, then each degree corresponds to $9 \times 8$ signals, and $9 \times 8 \times 181$ sets of data can be obtained. The 90 degree direction of the data emitted by No.0 array element and received by No.1 array element is selected to work as the experimental data, and the processing method of other angles is consistent with this. The time domain waveform of the data set is shown in figure 1.

![Fig. 1 The waveform of original signal in time domain](image1)

At first, principal component analysis is used to deal with the waveform obtained by the 90 degree direction of the phased array signal emitted by No.0 array element and received by No.1 array element. The sparse representation of the original signal is obtained, as shown in figure 2. It can be seen that the sparse coefficient of the phased array signal after PCA transform is mostly zero or close to zero, which is consistent with the characteristic of sparse signal.

![Fig. 2 The sparse coefficient after principal component analysis](image2)
Then, the length of ultrasonic phased array data is $N=1200$, and the number of observations $M=400$ is selected to complete the operation of signal projection observation, and the waveform shown in figure 3.

![Signal obtained by projection observation](image1.png)

**Fig. 3** The signal obtained by projection observation

Finally, the basis pursuit algorithm is used to deal with the ultrasonic phased array signal, and the reconstructed signal obtained are shown in figure 4.

![Signal reconstruction based on basis pursuit algorithm](image2.png)

(a) The reconstruction of the original signal

(b) The local enlargement of reconstructed signal

**Fig. 4** The signal reconstruction based on basis pursuit algorithm
5. Experimental error analysis

The reconstructed signal based on orthogonal matching pursuit algorithm has some differences in the signal waveform, compared with the original phased array signal. In order to analyze the effect of the reconstruction algorithm more accurately, the reconstructed error is displayed numerically, as shown in table 1. The absolute error $\Delta V$ and the relative error $\delta$ are calculated as below,

$$\Delta V = |V_0 - V_1|$$

$$\delta = \frac{|V_0 - V_1|}{V_1} \times 100\%$$

where $V_0$ is the amplitude of reconstructed signal at the point of maximum amplitude deviation, and $V_1$ is the amplitude of original phased array signal at the same point.

Table 1. Reconstruction error

<table>
<thead>
<tr>
<th>algorithm</th>
<th>BP</th>
</tr>
</thead>
<tbody>
<tr>
<td>The absolute error/V</td>
<td>0.1871</td>
</tr>
<tr>
<td>The relative error/%</td>
<td>0.39</td>
</tr>
</tbody>
</table>

where $|\Delta V| \leq 0.1871$, and $|\delta| \leq 0.39\%$. Then table 2 shows the error comparison of some common transform base and the principal component analysis method.

Table 2. The error comparison

<table>
<thead>
<tr>
<th>orthogonal transformation</th>
<th>PCA</th>
<th>DCT</th>
<th>DFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>The absolute error/V</td>
<td>0.1871</td>
<td>0.8943</td>
<td>21.0229</td>
</tr>
<tr>
<td>The relative error/%</td>
<td>0.39</td>
<td>1.88</td>
<td>1.27</td>
</tr>
</tbody>
</table>

where DCT is Discrete Cosine Transform, and DFT is Discrete Fourier Transform. From the analysis of experimental error, it indicates that the relative error is relatively lower than commonly used method. That is, the proposed method can be applied to signal sparse representation of compressive sensing.

6. Conclusion

This paper research on the compressive sensing sparse sampling method based on principal component analysis. This method not only solves the difficulty in storage and processing due to the large amount of data obtained by ultrasonic phased array structural health monitoring, but also effectively improves the relationship between the original signal and the signal after sparse representation. And the experimental result shows that principal component analysis can be used to reconstruct the signal obtained from the phased array structure health monitoring after sparse representation of the signal with small reconstruction error. In future research, we can choose more optimized projection observation matrix, and more efficient reconstruction algorithm to reconstruct the ultrasonic phased array signal more efficient.

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