Accurate Characterization of 3D Dispersion Curves and Mode Shapes of Waves Propagating in Generally Anisotropic Viscoelastic/Elastic Plates

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ABSTRACT

We investigate wave propagation in generally anisotropic viscoelastic plates. The generally anisotropic material system has 21 independent complex coefficients in its stiffness matrix when viscoelasticity is taken into account. The dispersion equation is obtained in the form of determinant vanishing in the complex domain based on the straightforward derivation. An accurate root-searching in the dispersion equation for arbitrary ranges of frequencies and complex wavenumbers, however, is very difficult, if not impossible. In this paper, a novel algorithm is introduced to calculate the 3D dispersion curves. An approximate solution in the low attenuation range via the semi-analytical finite element (SAFE) method is also used to compare and validate the introduced algorithm. Using this algorithm, various peculiar wave features are then investigated in details. These includes the attenuation jump and branches exchange in viscoelastic model caused by conversion of wave mode shapes, and the veering of dispersion branches in the corresponding elastic medium. The proposed accurate algorithm along with the observed features should be particularly useful in nondestructive evaluations via waves in viscoelastic/elastic plates and structures.

1. Introduction

Elastic waves are widely used in non-destructive evaluation (NDE) and structural health monitoring (SHM) technologies[1-4]. The characteristics of waves contain rich information about the structures, physical fields, and properties of the media, and in many cases, accurate calculation of dispersion curves is essential and imperative. As such, a robust and reliable method is essential and much desirable in characterizing wave features.

For some simple geometries, the dispersion equation can be derived exactly and various methods have been proposed to calculate the dispersion curves (or find the roots in the dispersion equation) associated with the given problems. The classical root-searching algorithm forms the basis of the software DISPERSE⁵,⁶ and it can deal with many wave propagation problems. However, some limitations on materials and geometries exist, including the case involving generally anisotropic materials with viscoelasticity⁷. When viscoelasticity is taken into account, the energies of the wave motion are dissipative, resulting in the appearance of complex values for all wavenumbers. As such,
the classical root-searching may fail since the searching needs to be carried out in the three-dimensional (3D) space, i.e. in the space made of real frequency and complex (real and imaginary) wavenumbers. This extra dimension makes the root-searching much more difficult than that in the corresponding two-dimensional (2D) space for the purely elastic case where only real or purely imaginary wavenumber is needed. Other root-searching algorithms, such as the method of bisection and Muller’s method\(^\text{[8]}\), are also with serious problems, not to mention the complicated dispersion relations due to the material anisotropy and the increased root-searching domain (real frequency vs. complex wavenumber).

Wave propagation (particularly dispersion curves) can be also studied directly via the numerical methods. The representative methods are: the orthogonal series method for Lamb wave propagation\(^\text{[9]}\) and for waves in multilayered piezoelectric plates\(^\text{[10]}\); the scaled boundary finite element method for plate structures\(^\text{[11]}\); the finite element method for viscoelastic laminated panels and cylinders\(^\text{[12, 13]}\); the spectral finite element (SFE) for linear viscoelastic laminates\(^\text{[14]}\); the spectral collocation method (SCM) for waves in generally anisotropic viscoelastic media\(^\text{[15, 16]}\); the semi-analytical finite element (SAFE) method for damped waveguides of arbitrary cross-section\(^\text{[17]}\), and for periodic structures\(^\text{[18]}\). While these methods are very flexible for irregular geometries\(^\text{[17, 19, 20]}\), accurate calculation of dispersion curves from dispersion equations is very difficult, if not impossible.

To accurately calculate the dispersion curves, we introduce in this paper the recently proposed root-searching algorithm\(^\text{[21]}\). Using this accurate algorithm, we then investigate waves in anisotropic viscoelastic/elastic\(^\text{[15, 16]}\). Various wave features are identified, and their similarity, difference, and connection are discussed based on the different material models. Some of our results extend and enhance some previous observations related to the interesting conversion of wave mode shapes\(^\text{[22]}\) along with veering of dispersion branches\(^\text{[23, 24]}\), whilst others are newly observed wave features.

2. General Anisotropic Materials, Damping Model and Dispersion Equation

For a general anisotropic elastic material (i.e., the triclinic one) and in terms of the Voigt notation, the constitutive relation between the stress \(\sigma_{ij}\) and strain \((0.5(u_{ij} + u_{ji}))\) can be expressed as

\[
\sigma_{ij} = c_{ijkl} u_{kl}
\]

where \(u_i\) are the displacements, and the subprime to them indicates the partial derivative with respect to the coordinates; the stiffness matrix \(c_{ijkl}\) has 21 independent components. Since the triclinic material has the most complicated crystal system, the present studies can be simplified to the other materials with less independent components in their stiffness matrices, particularly the isotropic material system. For the linear viscoelastic models, the viscoelasticity is defined by the corresponding complex stiffness matrix (the real elastic stiffness plus an imaginary damping stiffness). Two material damping models are considered -- the Kelvin–Voigt and hysteretic models\(^\text{[17]}\), as listed below, respectively, with the real elastic \(c_{ij}\) being replaced by.

\[
\begin{align*}
 c_{pq} &\leftarrow c_{pq} - i(f / \tilde{f}) \eta_{pq} \quad (2a) \\
 c_{pq} &\leftarrow c_{pq} - \eta_{pq} \quad (2b)
\end{align*}
\]

where \(f\) represents the frequency; \(i\) the imaginary unit. The viscosity tensor \(\eta_{pq}\) is measured at the fixed frequency value \(\tilde{f}\) for the Kelvin–Voigt model in Eq.(2a) and is independent of frequency for the hysteretic model in Eq. (2b). In these two models, \(\eta_{pq}\) has the same unit and symmetry as the elastic tensor \(c_{pq}\) shown in Eq. (1). It is noted that the two models are exactly the same at frequency \(\tilde{f}\).
We consider now the wave propagation within the plate and along $x_3$-direction (Fig. 1). Thus, the problem is reduced to the generalized plane-strain deformation with $\partial/\partial x_1=0$. As such, the equations of motion in terms of the (visco-) elastic displacements are deduced to

$$
\begin{align*}
&c_{46}u_{1,22} + 2c_{65}u_{1,23} + c_{55}u_{2,33} + c_{62}u_{2,22} + (c_{64} + c_{32})u_{2,23} + c_{34}u_{3,33} \\
&+ c_{64}u_{2,32} + (c_{35} + c_{54})u_{3,23} + c_{34}u_{3,33} = \rho \ddot{u}_1, \\
&c_{26}u_{1,22} + (c_{25} + c_{46})u_{1,23} + c_{45}u_{2,33} + c_{22}u_{2,22} + 2c_{24}u_{2,23} + c_{44}u_{2,33} \\
&+ c_{43}u_{3,23} + (c_{33} + c_{44})u_{3,33} + c_{34}u_{3,33} = \rho \ddot{u}_2, \\
&c_{46}u_{1,22} + (c_{46} + c_{56})u_{1,23} + c_{56}u_{2,33} + c_{42}u_{2,22} + (c_{44} + c_{32})u_{2,23} \\
&+ c_{41}u_{2,32} + c_{43}u_{3,22} + 2c_{42}u_{3,23} + c_{33}u_{3,33} = \rho \ddot{u}_3.
\end{align*}
$$

Due to the existence of 21 independent non-zero values in the stiffness matrix, the anti-plane displacement $u_1$ is coupled with the in-plane displacements $u_2$ and $u_3$ in Eq. (3). Physically, this means that the Lamb and SH waves observed in the isotropy case are coupled and cannot be separated from each other in the general case of triclinic materials. The boundary conditions are traction-free on the top and bottom surfaces. Making use of the constitutive relations Eq. (1), we have at $x_2=\pm h/2$,

$$
\sigma_{21} = 0, \quad \sigma_{22} = 0, \quad \sigma_{23} = 0. 
$$

Equations (3) and Eq. (4) are the governing equations and the boundary conditions respectively for the waves propagating in the triclinic plate. For a general time-harmonic wave solution of Eq. (3), we assume that

$$
\begin{align*}
u_1 &= A \exp(i\tilde{\xi}_2 x_2 + i\tilde{\xi}_3 x_3 - i\omega t), \\
u_2 &= B \exp(i\tilde{\xi}_2 x_2 + i\tilde{\xi}_3 x_3 - i\omega t), \\
u_3 &= C \exp(i\tilde{\xi}_2 x_2 + i\tilde{\xi}_3 x_3 - i\omega t).
\end{align*}
$$

where $\tilde{\xi}_2$ and $\tilde{\xi}_3$ are the wavenumbers along $x_2$- and $x_3$-directions, respectively, and $\omega=2\pi f$ is the angular frequency. Substituting Eq. (5) into Eq. (3) yields a set of linear equations for $A, B, C$. The determinant of the coefficients of $A, B, C$ has to vanish for nontrivial solutions which results in a sextic polynomial equation in $\tilde{\xi}_2$ for the given $\omega$ and $\tilde{\xi}_3$. There are six roots $\tilde{\xi}_2(m)$ ($m=1$ to 6) of this sextic equation with each yielding an independent nontrivial solution of Eq. (6). The amplitude ratios of $A, B, C$ then can be designated by $B(m)/A(m)=b(m)$ and $C(m)/A(m)=c(m)$, ($m=1$ to 6). Linear combination of these six independent nontrivial solutions provides the general solution of Eq. (3). Substituting the linear combination into Eq. (4), one obtains a system of linear equations in terms of $A(m)$ ($m=1$ to 6). Similarly, for the given $\omega$ and $\tilde{\xi}_3$, if the determinant of the coefficients of $A(m)$ vanishes, we then have a nontrivial solution which is a point on the dispersion curve. A new numerical algorithm proposed recently\textsuperscript{[21]} is adopted to attack it.

3. The New Method Proposed

The central idea of this method is briefly introduced here with the detailed processes and pseudocode being listed in reference\textsuperscript{[21]}. We consider a general transcendental equation $f(x)=0$ and replace the function $f(x)$ with its modulus $|f(x)|$. Then, $|f(x)|=0$ guarantees $f(x)=0$. Notice that since the value of
function $|f(x)|$ is always a non-negative real number, complex value of $f(x)$ can be avoided. A general graph of $|f(x)|$ is shown below.

To find the null points of equation $|f(x)|=0$, we propose the following new approach: First, we calculate all the minimal points of the function $|f(x)|$ including also the null points (due to $|f(x)|\geq 0$ as shown in Fig. 2). We then take the limit of $|f(x)|$ to these minimal points one by one. If a minimal point is not the solution, such as $m_0$ in Fig. 2, then approaching the minimal point $m_0$ from an arbitrary point $x=m_1$ around $m_0$, one obtains

$$\lim_{x\to m_0} \frac{|f(m_0)|}{|f(x)|} = \infty. \quad (7)$$

Therefore, if the minimal point is not a root, the modulus $|f(m_1)/f(x)|$ converges to a finite value $|f(m_1)/f(m_0)|$ due to $|f(m_0)|>0$. If it is a point on the dispersive curve, then the modulus $|f(n_i)/f(x)|$ approaches infinity due to $|f(n_0)|=0$. It is noticed that the magnitudes of $|f(m_1)|$ and $|f(m_0)|$ are close to each other since point $m_1$ is close to $m_0$, which ensures that the ratio $|f(m_1)/f(m_0)|$ is a finite value and is independent of the magnitude of $|f(m_0)|$. Thus, we can easily distinguish the values between the finite $|f(m_1)/f(m_0)|$ and the infinity so that the null points (the roots of the determinant) can be precisely identified for the dispersive curves. We emphasize that this novel root-searching approach is still applicable when the variable $x$ is a vector and $f(x)$ has a complicated expression (instead of polynomial form), provided that $|f(x)|$ can be calculated for any given $x$.

4. The Semi-Analytical Finite-Element Method

To validate the correctness of our analytical method, we can use the numerical methods instead of solving roots from exact dispersion equations. A common numerical method is the semi-analytical finite-element method (SAFE). In terms of the SAFE method, different from Eq. (5), the waves are written in the time-harmonic form only in the propagation direction $x_3$ as

$$u_j(x_1,x_2,x_3,t) = U_j(x_1,x_2) \exp(i\xi_j x_3 - io\omega t) \quad (8)$$

Substituting Eq. (10) into the displacement equations of motion $c_{ijn} u_{j,\alpha\beta} = \rho \ddot{u}_\alpha$, one obtains ($s=1,2,3$)

$$c_{ij,s} U_{j,\beta\alpha} + i(c_{ij,s} + c_{ijs}) \xi_j U_{j,\alpha} - c_{ij,s} \xi_j^2 U_j + \rho \omega^2 \delta_{ij} U_j = 0. \quad (9)$$
where $\delta_{ij}$ is the Kronecker symbol, and repeated indices take the summation from 1 to 2 ($\alpha, \beta$), or 1 to 3 ($t, j, n$); $i$ again represents the imaginary unit. Then, a finite-element solution of Eq. (11) under the boundary condition Eq. (4) can be solved numerically. A detailed description on generally anisotropic case using this numerical approach can be found in reference [18].

5. Numerical Results and Discussions

5.1 Material Parameters

The elastic and damping matrices can be found in reference [15]

\[
\begin{bmatrix}
74.29 & 28.94 & 5.86 & 0.20 & -0.11 & 37.19 \\
25.69 & 5.65 & 0.0928 & -0.0801 & 17.52 \\
12.11 & 0.0133 & -0.0086 & 0.22 \\
4.18 & 1.31 & 0.0949 \\
5.35 & -0.0705 & 28.29 \\
\end{bmatrix}
\]

\[\text{Sym.}\]

\[
\begin{bmatrix}
218 & 76.5 & 16.4 & -3.60 & 0.688 & 116 \\
71.1 & 19.2 & -0.771 & 2.15 & 50 \\
42.2 & -0.9644 & 0.627 & -3.07 \\
11.1 & 2.89 & -1.15 \\
13.6 & 1.48 & 93.5 \\
\end{bmatrix}
\]

For this material and geometric model, the results obtained from the present analytical method will be compared with those using the numerical SAFE method. It is noted that since the wavenumbers in the damping plate are complex, the waves with small attenuation are of particular interest as this means a large ratio between the real and imaginary wavenumbers.

5.2 Comparisons of Different Models and Different Methods

For a bulk wave traveling in a damping medium, its attenuation is defined as the loss per unit distance traveled, i.e., in the unit of Nepers (Np) per meter \(^{[25]}\), which is further equal to the imaginary wavenumber. Plotted in Fig. 3 is the attenuation and phase velocity vs. frequency for both the Kelvin-Voigt (a,c) and hysteretic (b,d) models, obtained using the present analytical method (red dots) and the numerical SAFE (blue circles).

\[\text{Figure 3. Attenuation vs. frequency } f \text{ for Kelvin–Voigt model (a) and hysteretic model (b); Phase velocity vs. frequency } f \text{ for Kelvin–Voigt model (c) and hysteretic model (d); The present analytical}\]
method (red dots) vs. the numerical SAFE method (blue circles) with attenuation range of 0-100 Np/m (or imaginary wavenumber $\text{Im}(\xi_3)$ from 0-0.1 mm$^{-1}$). Points A in (a) and B in (b) correspond to those in (c) and (d).

It is observed clearly from Fig. 3 that the results of both attenuation and phase velocity based on the analytical solution and the numerical SAFE agree well with each other, demonstrating the correctness of the present analytical method. It is noted that the attenuations are the same when frequency $f=2\text{MHz}$ due to Eq. (2). It is observed that Figs. 3c and 3d are nearly identical to each other, except for some curves where they end at different phase velocities. Should the attenuation be increased further the curve ending at point A in Fig. 3c would extend to point B in Fig. 3d, which actually shows no difference in real wavenumbers. Further, accurate searching for the complex roots is critical for characterizing correctly the wave features involved. This is presented in detail below.

5.3 3D Dispersion Curves

After validating our analytical solutions against the numerical SAFE results, we now analyze in detail the 3D dispersion curves and the associated new features on some branches of the curves. We first point out that Eq. (6) is a quadratic function of both $\xi_2$ and $\xi_3$, whilst each expression in the boundary condition Eq. (4) is homogeneous function of $\xi_2$ and $\xi_3$. Thus, for a given frequency $f$, if $\xi_3$ is a root of the determinant (i.e., a point on the dispersive curve), so is $-\xi_3$. This feature of the dispersion equation renders the well-known central symmetry of dispersion curves$^{[16]}$. As such the 3D dispersion curves can be characterized completely in terms of the positive imaginary wavenumber vs. frequency only; the negative imaginary part can be simply obtained by rotating the curves 180 degrees around the frequency axis.

![Figure 4. 3D dispersion curves: Kelvin–Voigt model in (a) and hysteric model in (b).](image)
The 3D dispersion curves for both Kelvin–Voigt and hysteretic models are shown in Figs. 4a and 4b respectively. It is noticed that, when frequency \( f \) varies from 0 to 5 MHz, the imaginary wavenumber Im(\( \xi_3 \)) varies from 0-10 mm\(^{-1}\); however, the variation of the real wavenumbers Re(\( \xi_3 \)) has no limit. In other words, the dispersion curves for these two viscoelastic models are mainly distributed along the (positive) Re(\( \xi_3 \))-axes.

Such waves with low attenuation are actually very useful in non-destructive evaluation since they can propagate a large distance and, therefore, can further differentiate themselves from noise\(^{[2]}\) due to their slow decaying. We compare them with the purely elastic model below for an in-depth investigation.

![Figure 5](image)

**Figure 5.** 2D dispersion curves of frequency \( f \) vs. real wavenumber Re(\( \xi_3 \)) (or wavenumber \( \xi_3 \)):
Kelvin–Voigt model in (a), and the corresponding purely elastic model in (b) (i.e., \( \eta_{pq}=0 \) in the Kelvin-Voigt model). Marked boxes 1-6 are for detailed analysis below, and marked ellipses 1-4 are for comparison.

Shown in Figs. 5a and 5b are the 2D dispersion curves for both the Kelvin-Voigt model and its corresponding purely elastic model. The 2D dispersion curve from the hysteretic model is almost the same as in Fig. 5a. Comparing Fig. 5a to 5b, the effect of material damping is clearly demonstrated, particularly near the region where Re(\( \xi_3 \)) is around zero. The curves in the ellipses 1-4 of Fig. 5b for the corresponding purely elastic case disappear in the same locations in Fig. 5a for the Kelvin-Voigt damping model. Interestingly, however, these parts still exist with attenuation (i.e. Im(\( \xi_3 \)) <0; thus, they are invisible in the side view of Fig. 4a with Im(\( \xi_3 \)) >0 but can be clearly observed in the negative parts of Re(\( \xi_3 \)) in Fig. 5a due to the central symmetry of the dispersion curves about the frequency \( f \) axis as mentioned above. On the other hand, introducing damping could cause the disappearance of cut-off frequencies observed in the purely elastic model (as also observed for the isotropic case by reference\(^{[24]}\)). Besides these regions with obvious differences, in other locations, the wave features between the viscoelastic and purely elastic materials can be sharply different as well. To demonstrate
this, we have identified six locations where the waves on different modes are either very close to each other or crossover (overlapping) with each other. These particular locations are boxed as 1 to 6. Curves in these boxes look like the same when comparing Fig. 5a to Fig. 5b; however they can be completely different as analyzed in detail below.

5.4 Peculiar Features of Attenuation and Branch Switch

As observed above, the curve features in the six boxes in Fig. 7 can be completely different for both viscoelastic and purely elastic models. To demonstrate this, we select only boxes 1 and 2 for comparison between the two models.

![Figure 6. Zoom-in dispersion curves between Kelvin-Voigt model and its corresponding purely elastic model in box 1 of Figs. 5a and 5b. Purely elastic model with real wavenumbers in black, and Kelvin-Voigt model with complex wavenumbers in red and blue.](image)

The zoom-in dispersion curves in box 1 for both Kelvin-Voigt model and its corresponding purely elastic model are shown in Fig. 6. It is observed from Fig. 6a that the two branches (curves) in purely elastic case are clearly separated but those in the Kelvin-Voigt model looks intersected with each other. However, looking at them from the 2D front view in Fig. 6b, there is no intersection for the curves in box 1, just like the case in box 6. In other words, the curves of Kelvin-Voigt model overlap these of the purely elastic model completely. Figure 6c (2D side view of \( f \) vs. Im(\( \xi \))) indicates that on these two branches, while Im(\( \xi \)) is identically zero for the purely elastic case, Im(\( \xi \)) varies over a large finite interval (roughly from 3 m\(^{-1}\) to 8 m\(^{-1}\)) for small variation of frequency. We call this “unstable” feature associated with damping.
Comparing Figs. 7c,d to Figs. 6c,d, it is noted that on these two branches with varying $f$ or $\text{Re}(\xi)$, we now have either $\text{Im}(\xi)$ is zero (purely elastic case) or at nearly fixed values (i.e., nearly vertical line or “stable” in Kelvin-Voigt model). It is observed that while the two branches are still separated from each other in the purely elastic case, they switch from each other after passing box 2 for the Kelvin-Voigt model, as compared to the corresponding purely elastic model. Similar features shown in box 2 can be observed for the curves (or branches) in boxes 3-5 and they all indicate that damping can substantially alter the dispersion behaviors as compared to the corresponding purely elastic case.

### 5.5 Mode Conversion, Branch Veer, Attenuation Jump, and Branches Exchange

In this subsection, we consider in detail the conversion of wave mode shapes and veering of dispersion branches in the elastic model, attenuation jump and branch exchange in the corresponding viscoelastic model. We further investigate the correlation between them.
Figure 8. The first five branches for both Kelvin–Voigt model and its corresponding purely elastic model: Frequency vs. wavenumber for purely elastic model in (a), group velocity vs. frequency for purely elastic model in (b), frequency vs. real wavenumber for Kelvin-Voigt model in (c), attenuation vs. frequency for Kelvin-Voigt model in (d). Boxes 1, 2 and 6 are further marked by sampling points A, B, C, and D in (a) and (b) for purely elastic model. The sampling points a, b, c, d which have the same real wavenumber and frequency as A, B, C, D respectively (comparison with Figs. 5a and 5b) are also marked correspondingly in (c) and (d) for Kelvin-Voigt model.

Figure 9. Wave mode shapes at sampling points A1, B1, C1, and D1 in box 1 of Figs. 8a and 8b for the purely elastic case. Location 1 and Location 2 have a difference of a quarter of wave length along wave propagating direction \( x_3 \), i.e. \( x_{3\text{Location 2}} - x_{3\text{Location 1}} = \pi/(2\xi_3) \).

It is noted that for purely elastic model in Fig. 8a, obvious veering occur for dispersion branches in boxes 1, 2, and 6. For a better observation, group velocity \( (V_g=d\omega/d\xi_3) \) vs. frequency for the purely elastic model is shown in Fig. 8b where the group velocity jump s correspondingly due to veering. For the purely elastic model, the wave mode shapes in these veer regions are also calculated as shown in Fig. 9 for points A1, B1, C1, and D1. The results show that A1 and B1 are on the same branch (branch 3) in Fig. 8a; however, they have different wave mode shapes as shown in Fig. 9. On the other hand, A1 and C1 are on branch 3 and branch 2 respectively, but they have similar wave mode shapes due to the similar group velocity in Fig. 8b. The same situations happen in all the sampling points A, B, C, and D in boxes 1, 2, and 6. These phenomena of wave mode shapes conversion along the same branch and veer of different dispersion branches were also observed and discussed for the corresponding isotropic material models\(^{[22-24]}\). It is noted that the branch 1 in Fig. 8a varies very smoothly with nearly a constant group velocity shown in Fig. 8b, indicating further no rapid changes in the wave mode shapes. In this case, when Kelvin–Voigt viscoelastic model is considered, the attenuation for branch 1 shown in Fig. 8e is a quadratic function of frequency. This quadratic relation between attenuation and frequency is typical for Kelvin–Voigt model\(^{[17, 25]}\). We consider again the regions with conversion of wave mode shapes and veering of branches in the purely elastic model, such as box 1. In the corresponding Kelvin–Voigt viscoelastic case, points b1 and d1 with similar wave mode shapes in the purely elastic case have their attenuation close to each other as shown in Fig. 8e. The attenuation of points b1 and d1 still has small difference which is caused by the quadratic relation between attenuation and frequency. However the attenuation at points a1 and b1 has a larger difference due to the different wave mode shapes in A1 and B1 although they are on the same branch (branch 3). Naturally, a jump in attenuation occurs on branch 3 from a1 to b1 in Fig. 8e (refer also to 3D curves in Fig. 6). This change in attenuations for sampling points a, b, c, and d is more obvious in boxes 6 and 2 at higher frequency. Especially in box 2, points a2 and b2 have a very large difference in their attenuation in Fig. 8e. This results in the branches exchange compared to the purely elastic mode case (refer also 3D curves in Fig. 7), rather than attenuation jump shown in boxes 1 and 6. While the discussion above is based on the Kelvin–Voigt model and its corresponding purely elastic one, similar features can be observed in other viscoelastic models such as the hysteretic model. The only difference is that, if the group velocity changes only slightly in its corresponding purely elastic case, the attenuation for the hysteretic model is a linear function of frequency rather than a quadratic one, as can be observed from Figs. 3a and 3b above\(^{[17, 25]}\).

6. Conclusions
In this paper, dispersion equations of a generally anisotropic plate (triclinic) are derived. Based on a newly proposed algorithm, 3D dispersion curves for two viscoelastic (Kelvin-Voigt and hysteretic) models are calculated and compared. These viscoelastic results are further compared to the corresponding elastic model. The group, branches trend, and wave mode shapes are all discussed in detail.

1. The 3D dispersion curves with low attenuation for the two viscoelastic (Kelvin-Voigt and hysteretic) anisotropic models are concentrated in the regions where their real wavenumber and imaginary wavenumber are both positive. Different viscoelastic models result in significant differences in their wave features with low attenuation; the difference, however, is small on waves with high attenuation.

2. The cut-off frequencies observed in purely elastic model disappear in the corresponding viscoelastic model. At the same time, compared to purely elastic model, some branches are invisible near the frequency axis in viscoelastic model. This is due to the fact that the curves with positive real wavenumbers near the cut-off frequencies in purely elastic case gain a negative attenuation by damping, which results in the disappearance of dispersion curves in the corresponding viscoelastic model.

3. Conversion of wave mode shapes and veering of branches (in relatively lower-order) exist in neighboring branches in purely elastic triclinic plate. When viscoelasticity is taken into account, the attenuation is correlated to the wave mode shapes rather than to the dispersion branch. Therefore, attenuation jump or branch switch occurs in viscoelastic model due to conversion of wave mode shapes in these regions. On the other hand, for similar wave mode shapes, attenuation changes according to the characteristics of different viscoelastic models (quadratic function of frequency in Kelvin-Voigt models, and linear function of frequency in hysteretic model).

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References


