Perturbation methods for analysis of the free vibration of thin plates with oblique crack

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ABSTRACT

This paper proposes an analytical perturbation method for the analysis of the free vibration of thin plates with oblique cracks. The crack is modeled as a slight reduction in the plate thickness for the simulation of early damage in the existing plate-like structures. Based on the above cracked plate model, the effects of main characteristic parameters of the crack involves direction, length and location on the natural frequency and the derivation of mode shapes like curvature mode shapes are discussed. Results are presented to demonstrate different sensitivity and applicability of the derivation of mode shape with respect to different low level crack cases. The chief aim of this paper is to provide an effective platform to establish crack scenarios for the assessment and parameter optimization of any damage identification algorithm.

1. Introduction

Thin-walled structures in the form of plates are often encountered in civil, aeronautical, marine engineering structures. The initial cracks in plates maybe initiated under cyclic loads or surrounding-induced material aging and the orientation of cracks are random. As a basis of dynamics-based SHM technique, analytical formulations for the dynamic response of a plate with a variably orientated crack are of great importance. Huang and Leissa[1] first developed a special displacement function consisting of the well-known admissible function of the Ritz method and corner functions inspired by Williams' asymptotic solutions to describe the stress singularities at the crack tip and the discontinuities of displacement and slope crossing crack for the vibration of rectangular plates with side cracks at different locations with various lengths and orientations. And after that, Huang et al.[2] further applied the above method to analyze the free vibrations of rectangular plates with internal cracks at different locations with various lengths and orientations. Also, Huang et al.[3] expanded the work to accurately determine the frequencies and nodal patterns of thick, cracked rectangular plates based on the Mindlin plate theory. The slanted cracks in the above literatures are the penetrating cracks, however for the most cracks in their initial state are the part-through. Rice and Levy[4] modeled crack shape as a semi-ellipse, and crack mechanical properties are abstracted to a line-spring having both stretching and bending resistance. Compliance coefficients at points along the line-spring are matched to those of the edge cracked strip in plane strain. Kuo et al.[5] used an accurate three-dimensional analysis called the body method to further demonstrate the validation of above part-through crack simplification when crack length is wider than their depth. In order to simplify
calculating for approximate engineering, King\textsuperscript{[6]} proposed a simplified line-spring model by replacing the crack front with a crack of constant depth and treating the ligament spring as elastic perfectly plastic. Typically, when the crack is away from the centre of thin plate, the line spring model proposed is not practicable. So, Yu and Jin\textsuperscript{[7]} modified the compliance coefficient at points along the line-spring by using a Modified Line Spring Model (MLSM). Different from the above-mentioned motion of equation based on the large-deflection theory which involved the membrane forces and the compliance coefficients at crack location which involved the stress intensity factor, Sharma et al\textsuperscript{[8]} modeled crack as a localized reduction in the plate thickness and analyzed the free vibration in a rectangular thin plate with a proposed crack paralleled to the plate edge relied on the small-deflection theory and the compliance coefficients at crack location is simplified equaled to the compliance coefficient of thickness-reduced thin plate. The present work extended the free vibration formulation presented in Sharma et al\textsuperscript{[9]} to a simple-supported rectangular thin plate with a part-through surface crack of arbitrary orientation and position which stress concentration effect is negligible.

The rest of paper is organized as follows. The perturbation method for analysis of the free vibration of cracked thin plates is introduced in Section 2. Based on above cracked plate model, the effects of main characteristic parameters of the crack involves direction, length and location on the modal parameters including natural frequency and the commonly used curvature mode shapes for thin plate are discussed in Section 3. Finally, Section 4 summarized the main results of the work.

2. Analytical Cracked Plate Model

2.1 Free Vibration of a Simply Supported Plate with an Arbitrarily Orientated Part-through Long-narrow Crack

The free vibration of a simply supported plate at all edge of length $L_x \times$ width $L_y \times$ thickness $h_0$ with a long-narrow rectangular crack defect which has arbitrarily orientation $\theta$ and part-through depth of $h_D$ as shown in Figure 1 is considered. Dynamic behavior of locally cracked plates can be formulated from the general equation of motion for plates of variable thickness by Leissa \textsuperscript{[9]}:

\[
\nabla^2 (D \nabla^2 w) - (1 - \nu) \left( \frac{\partial^2 D}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 D}{\partial y \partial x} \frac{\partial^2 w}{\partial y \partial x} + \frac{\partial^2 D}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right) + m \frac{\partial^2 w}{\partial t^2} = 0
\]

(1)

where $w=w(x, y)$ is the out-of-plane displacement of the plate, $h=h(x, y)$ is the plate thickness, $D=D(x, y)=Eh^3/12(1-\nu^2)$ is the plate flexural rigidity, and $m=m(x, y)=ph(x, y)$ is the mass per unit area of the plate. $E$, $\rho$, and $\nu$ are the Young’s modulus, density, and Poisson’s ratio of the plate, respectively.
Figure 1. Schematic of plate with Arbitrarily Orientated part-through crack defect

By assuming the displacement function based on modal superposition theory as:

\[ w(x, y, t) = \sum_{i,j} \phi_{i,j}(x, y)e^{i\omega t} \] (2)

where \( \phi_{i,j}, w_{i,j} \) are, respectively, the \((i, j)\) mode shape and natural frequency of the plate, while \( I \) is the imaginary. Considering a single mode \((i, j)\) and substituting Eq. (2) into Eq. (1) give:

\[ \nabla^2(D\nabla^2\phi) - (1-\nu)\left( \frac{\partial^2 D}{\partial x^2}\frac{\partial^2 \phi}{\partial x^2} - 2\frac{\partial^2 D}{\partial y \partial x}\frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 \phi}{\partial y^2} \right) - m\lambda \phi = 0 \] (3)

where \( \lambda = \omega^2 \), and the subscript \((i, j)\) is omitted for simplicity.

The plate rigidity distribution function \( D(x, y) \) can be formulated as:

\[ D(x, y) = D_0[1 - \varepsilon Lb \delta(x \cos \theta + y \sin \theta - \rho)] \] (4)

where \( \delta(\rho) = \frac{dH(\rho)}{d\rho} \) is the Dirac delta function, \( b_0 \) is the oblique crack width and \( \varepsilon \) is a minimal value. The crack range parameter \( L \) can be expressed as:

\[ L = \int_{\xi}^{\eta} \delta(x - \xi)d\xi \int_{\eta}^{\eta} \delta(y - \eta)d\eta \] (5)

where \( \xi \) and \( \eta \) are dummy integration variables; \((x_1, y_1)\) and \((x_2, y_2)\) are the coordinates of the crack endpoint in the coordinate system shown in Figure 1.

The mass per unit area distribution of damaged plate can be described as:

\[ m(x, y) = m_0[1 - \varepsilon Lb \delta(x \cos \theta + y \sin \theta - \rho)] \] (6)

With the assumption that the damage magnitude parameter, \( \varepsilon \), is small, the perturbation method to the undamaged plate can be used to solve the governing equation (Eq. (3)) for the damaged plate. Detailed derivation process is recommended to refer to Shi and Qiao\(^{10}\).

The eigensolutions of the damaged plate can be expressed as:

\[ \lambda = \lambda_0 - \varepsilon \lambda_1 \] (7)

\[ \phi(x, y) = \phi_0(x, y) - \varepsilon \phi_1(x, y) \] (8)

where \( \lambda_0 \) and \( \phi_0(x, y) \) are the eigensolutions of undamaged plate; while \( \lambda_1 \) and \( \phi_1(x, y) \) are the first order perturbation.

The eigenvalue changes \( \lambda_{pq}^{pq} \) and the coefficient of mode shape changes \( \eta_{pq} \) can be, respectively, expressed as:

\[ \lambda_{pq}^{pq} = \frac{4Dib_0}{m_0L_sL_y} \beta_1 \] (9)

\[ \eta_{pq} = \frac{4b_0}{L_sL_y} \frac{-\beta_1 + (1-\nu)\beta_2}{((\frac{\pi}{L_s})^2 + (\frac{\pi}{L_y})^2)^2 - ((\frac{i\pi}{L_s})^2 + (\frac{i\pi}{L_y})^2)^2} \] (10)

where

\[ \beta_1 = G_1 + G_2 + G_3 + G_4 + G_5 \] (11)

\[ \beta_2 = M_1 + M_2 + M_3 \] (12)
where

\[ G_1 = \int_{\Gamma} \left[ \frac{2}{3} \left( \frac{i \pi}{L_x} \right)^2 \sin \frac{i \pi x}{L_x} \sin \frac{j \pi y}{L_y} \sin \frac{r \pi x}{L_x} \sin \frac{s \pi y}{L_y} \right] dt \]

\[ G_2 = \int_{\Gamma} \frac{-2i(j^2 L_y^2 + i^2 L_x^2)\pi^4 \sin \frac{j \pi x}{L_x} \sin \frac{s \pi y}{L_y} \sin \frac{r \pi x}{L_x} \cos \frac{i \pi x}{L_x} \sin \frac{r \pi x}{L_x} \sin \frac{i \pi x}{L_x}}{L_y^2 L_x} dt \]

\[ G_3 = \int_{\Gamma} \frac{-2j(j^2 L_y^2 + i^2 L_x^2)\pi^4 \sin \frac{i \pi x}{L_x} \sin \frac{r \pi x}{L_x} \sin \frac{r \pi x}{L_x} \cos \frac{s \pi y}{L_y} \cos \frac{j \pi y}{L_y} - j \sin \frac{s \pi y}{L_y} \sin \frac{j \pi y}{L_y}}{L_y^2 L_x} dt \]

\[ G_4 = \int_{\Gamma} \frac{-(j^2 L_y^2 + i^2 L_x^2)\pi^4 \sin \frac{s \pi y}{L_y} \sin \frac{j \pi y}{L_y} \sin \frac{r \pi x}{L_x} \cos \frac{i \pi x}{L_x} \sin \frac{r \pi x}{L_x} \sin \frac{i \pi x}{L_x}}{L_y^2 L_x} dt \]

\[ G_5 = \int_{\Gamma} \frac{-(j^2 L_y^2 + i^2 L_x^2)\pi^4 \sin \frac{i \pi x}{L_x} \sin \frac{r \pi x}{L_x} \sin \frac{r \pi x}{L_x} \cos \frac{s \pi y}{L_y} \cos \frac{j \pi y}{L_y} - (r^2 + i^2) \sin \frac{r \pi x}{L_x} \sin \frac{i \pi x}{L_x}}{L_y^2 L_x} dt \]

and,

\[ M_1 = \int_{\Gamma} \frac{i^2 \pi^4 \sin \frac{r \pi x}{L_x} \sin \frac{i \pi x}{L_x} [(s^2 + j^2) \sin \frac{s \pi y}{L_y} \sin \frac{j \pi y}{L_y} - 2sj \sin \frac{s \pi y}{L_y} \cos \frac{j \pi y}{L_y}]}{L_y^2 L_x} dt \]

\[ M_2 = \int_{\Gamma} \frac{j^2 \pi^4 \sin \frac{s \pi y}{L_y} \sin \frac{i \pi x}{L_x} [(r^2 + i^2) \sin \frac{r \pi x}{L_x} \cos \frac{i \pi x}{L_x} - 2r \cos \frac{r \pi x}{L_x} \cos \frac{i \pi x}{L_x}]}{L_y^2 L_x} dt \]

\[ M_3 = \int_{\Gamma} \frac{2ij \pi^4 \sin \frac{s \pi y}{L_y} \sin \frac{j \pi y}{L_y} - j \sin \frac{s \pi y}{L_y} \sin \frac{j \pi y}{L_y} \sin \frac{r \pi x}{L_x} \sin \frac{i \pi x}{L_x} - r \cos \frac{i \pi x}{L_x} \cos \frac{r \pi x}{L_x}}{L_y^2 L_x} dt \]

where \( \Gamma \) is the line segment determined by, \( \Gamma : x = t \cos \theta + x_0, y = t \sin \theta + y_0 \), where \((x_0,y_0)\) is the center of crack line.

3. Panorama on possible crack damage

In this section, numerical results of the modal parameter include natural frequency and mode shape are presented for the aboved plate model. The type of material used in this study is an aluminum alloy of 6061-T6, with the following material properties: Young’s modulus \( E=70.3 \) GPa, plate density \( \rho=2700 \) kg/m\(^3\), Poisson’s ratio \( \nu=0.33 \). Firstly, the crack scenarios with respect to the location, length and orientation are considered to set up a panorama of the influence of on natural frequency, as well as a pilot study to frequency-based damage identification possibility for tiny damage. Secondly, studies are presented for curvature mode shape of the cracked plate model with identical scenarios. The manifestation pattern of long narrow crack in different location provide the theoretical direction for the choice of parameters in the subsequent curvature mode shaped-based damage identification technology.

3.1 Influence of directional crack on the natural frequency
The geometry of the plate used in this section is defined as: square plate \((L_x=0.4\, \text{m}, L_y=0.4\, \text{m})\) with plate thickness, \(h_0=0.005\, \text{m}\). The width \(b_\theta\) of long narrow crack is set as 0.001m. Table 1 presents the \((1,1)\)th natural frequency change ratio (NFCR) between the intact square plate and corresponding cracked plate with crack relative depth \((h_D/h_0=0.09)\), different length of half-crack \((a=0.005\, \text{m} \text{ and } 0.01\, \text{m})\) at different location and for different values of orientation angle, \(\theta\). The orientation angle is varied from 0° to 165°, in 15° steps. The crack center location \((x_c, y_c)\) is selected at three representative place of mode shape: longitudinal or transverse nodal line \((0.01\, \text{m}, 0.2\, \text{m})\), intersection of longitudinal and transverse nodal line \((0.01\, \text{m}, 0.01\, \text{m})\) and crest or trough \((0.2\, \text{m}, 0.2\, \text{m})\).

The effects of main characteristic parameters of the long narrow crack on the natural frequency are discussed as following: (1) crack length. As shown in Table 1, the NFCR of square plate proportionately increases as crack length increases. (2) crack direction and location. More intuitive comparison of square plate is shown in the Figure 2. When crack is located at intersection of longitudinal and transverse nodal line \((0.01\, \text{m}, 0.01\, \text{m})\), the maximum NFCR occurs when the crack direction is consistent with the diagonal direction of the plate, while the minimum NFCR occurs when the crack direction is perpendicular to the diagonal direction. When crack is located at longitudinal or transverse nodal line \((0.01\, \text{m}, 0.2\, \text{m})\), as expect, the maximum NFCR occurs when the crack direction is perpendicular to the nodal line, while the minimum NFCR occurs when the crack direction is parallel to the nodal line.

All told in Table 1, the natural frequency is extremely insensitive to the tiny crack with all cases, which imposes restrictions on the practical application of frequency-based damage identification technology. Moreover, as shown in Figure 2, different direction or location cause the same frequency change. As a result, false positives and false negative of identification is unavoidable.

Table 1. Natural frequency change ratio between the intact square plate and cracked models with a variably orientated surface long narrow crack for simply supported (SSSS) boundary conditions, at various orientation angles and various typical locations

<table>
<thead>
<tr>
<th>crack angle, (\theta) (deg.)</th>
<th>First, natural frequency change ratio, (\text{NFCR}_{ij}) (%(\text{oo}))</th>
<th>Location of crack line center ((x_c, y_c))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Intact plate (L_1=0.4, \text{m})</td>
<td>Cracked plate, (h_D/h_0=0.09)</td>
</tr>
<tr>
<td></td>
<td>(L_2=0.4, \text{m})</td>
<td>(a=0.005, \text{m})</td>
</tr>
<tr>
<td>Frequency((\text{Hz}))</td>
<td>((0.01,0.01))</td>
<td>((0.01,0.2))</td>
</tr>
<tr>
<td></td>
<td>((0.2,0.2))</td>
<td></td>
</tr>
<tr>
<td>153.1940261</td>
<td>-</td>
<td>-</td>
</tr>
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<td>0.6607851</td>
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</tr>
<tr>
<td>165</td>
<td>0.1030715</td>
<td>0.6607851</td>
</tr>
</tbody>
</table>
cracked plate, $a=0.01(m)$

<table>
<thead>
<tr>
<th>Angle (°)</th>
<th>$\text{Nat. Freq.}_x$</th>
<th>$\text{Nat. Freq.}_y$</th>
<th>$\text{Nat. Freq.}_z$</th>
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<tbody>
<tr>
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<td>1.3153060</td>
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</tr>
</tbody>
</table>

Fig 2. The (1,1)th normalized NFCR of cracked plate with different crack direction at three representative place of mode shape

3.2 Influence of directional crack on the mode shape

Many researchers have proved that curvature mode shape is more sensitive to tiny damage either in beams or plates than corresponding displacement mode shape\[^{[11]}\]. However, the lack of systematic investigation on the choice of the curvature for directional damage is a prominent problem, which hinders the effective application of curvature-based damage identification. In this section, the identical crack scenarios in natural frequency cases are considered in the curvature mode shape. The length of crack is fixed at 0.01m and the relative depth is fixed at 9%. Fig. 3 exhibits the characteristics of $\kappa_{xx}$, $\kappa_{yy}$ and $\kappa_{xy}$ of square plate with different crack directions at crack location of (0.2,0.2)m. The curvature mode shape considered in Fig. 3 is at (1,1)th natural frequency. The curvature mode shape only reveals the square region of 0.05m $\times$ 0.05m where the crack is.
The characteristics of $\kappa_{xx}$, $\kappa_{yy}$, and $\kappa_{xy}$ of square plate with different crack directions at typical crack positions of (0.2,0.2)m.

The single red lines in Fig. 3 represent the actual crack location. The quality of crack representation using the local eccentricity of curvature mode shapes basically depends on length representation and direction representation. In terms of the square plate, the following pattern can be seen from Fig 3: when the angle between the crack direction and $\kappa_{xx}$ direction changes from parallel to vertical, the crack representation effect changes from poor to good. The law of crack representation in $\kappa_{yy}$ is just the same as $\kappa_{xx}$. $\kappa_{xy}$ is very poor to represent the crack which is parallel to the plate edge, while non-parallel plate edge crack is good.

4. Conclusion

The free vibration behavior of damaged thin plates is established through perturbation techniques. The analytical plate model allows inspecting the effect of different kinds of damage on the plate modal parameters. The formulation takes oblique long-narrow defect into account which can be considered as the approximate descriptions of cracks in service structures. Variation of natural frequency and curvature mode shape are investigated for classical crack scene. The results imply that the low sensitivity of natural frequency and the high sensitivity of curvature mode shape to the presence of crack. Also, the directivity and size of crack limits the universality of the curvature mode shape. The presented analytical results provide comprehensive insights on the optimal selective of curvature mode.
shape for the assessment and parameter optimization of any curvature-based damage identification algorithm.

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References and Footnotes