IDENTIFICATION OF STIFFNESS DISTRIBUTION AND DAMAGE IN EULER-BERNOUILLI BEAMS USING STATIC RESPONSE

L. Li, F. Ghrib and W. Polies
Department of Civil and Environmental Engineering, University of Windsor, On, CANADA
fghrib@uwindsor.ca

ABSTRACT

The paper presents two computational inverse procedures for reconstructing the stiffness distribution and for detecting damage in beams using static responses. The complete deflection profile is obtained by processing digital images of the beam and it is utilized as the input for the inverse computational procedures. The first procedure formulation is based on the equilibrium gap through Euler-Bernoulli beam finite element discretization; the second is formulated as a minimization a data discrepancy functional. Examples of a simply-supported beam are simulated to demonstrate the performance of the two techniques. Numerical simulations show that the equilibrium gap method has good performance in locating and quantifying local damage and data discrepancy minimum functional is better used for reconstructing distributed global stiffness. Both two techniques can be integrated to be used as general identification of randomly distributed damage.

Keywords: Equilibrium gap method, adjoint optimization, Inverse problem, Damage identification, Beam stiffness.

INTRODUCTION

The identification of damage in beams have attracted intensive research efforts due to the increasing demands for assessing integrity and reliability of structures in service. Damage accumulation in structures can be associated to stiffness reduction, and expressed mathematically as a stiffness factor or a damage variable related to the stiffness [1-2]. The damage is naturally related to the change in the internal structure of materials within the cross-section of the beam. Damage identification is usually performed based on experimental data of actual beams subjected to external loadings.
Depending on the response data collected, damage identification methods can be divided into two categories: dynamic based and static based. The former category employs dynamic characteristics such as natural frequencies [1][3], displacement modal shapes [4][5], modal shape derivatives [6][7], wavelet analysis of dynamic signals [8], harmonic responses [2][9-10]. A recent overview of dynamic based methods is given in [11]. The literature on static response-based identification is much more limited. Banan et al gave a formulation estimating the constitutive properties of a finite-element model from measured displacements under a known static loading through minimizing the error in displacements at the measurement sites [12]. Other works include locating crack through wavelet analysis of the static deflection profile [13], using strain integral [14], and single or double crack identification from static loading and displacement measurements at several points [15].

Another way to differentiate identification methods is whether the problem is to locate and quantify damage in the form of simple discrete cracks, or to completely reconstruct a spatially distributed stiffness. Most of the damage detection methods aim at finding the location and quantifying localised damaged zones [14-17]. In this category of approaches, cracks are usually parameterized beforehand, which assumes knowledge of stiffness distribution of the structure. For methods dealing with distributed stiffness, the problem consists at recovering a continuous distribution of the beam’s stiffness. Mathematically, this problem involves a minimisation of a functional, and it is generally a more difficult inverse problem. Usually, the identification of distributed stiffness requires much more information collected testing of the beam. For example, Liu and Chen [9], Kokot and Zembaty [10] used harmonic response at a wide frequency range and spatially distributed points (all the nodes of a finite element model) as input for stiffness reconstruction of beams. Procedures efficient in localizing and quantifying concentrated cracks usually assumes a good knowledge of stiffness distribution of the undamaged structure, while in reconstructing stiffness distribution the damage is assumed continuous across the beam. Numerical algorithms performing well in one method may be inefficient for the other.

Recent progresses in image processing techniques and digital cameras make it possible to measure continuous deflection of structures under static loading from digital images [18]. This progress offers new possibilities in structural parameter identification and damage detection and localization. Unlike point-based testing sensors (displacement gauge, strain gauge, LVDT), digital images (Digital Image Correlation and photogrammetry) are able to provide measurement at a large number of spatially distributed points. The quasi continuous measurement of the beam deflection is more valuable as input for inverse procedure leading to damage detection and health monitoring of structures.

In the present paper, two inverse computational techniques are discussed. One is based the equilibrium gap method formulated on the basis of the equilibrium of internal forces; and the second method is based on a data discrepancy functional to minimize the difference between measured and simulated deflection profile. The total variation (TV) regularization technique is used to overcome the ill-posedness inherent in this type of inverse problems. Examples of a simply-supported beam are calculated showing the performances of the presented two methodologies.
EQUILIBRIUM GAP METHOD

Claire et al [19] presented a formulation based on the equilibrium gap to identify damage in 2D structures. In case of 2D elasticity and in the absence of volumetric loading away from the boundary, the continues format of the equilibrium equations is \( \text{div}(\sigma(E, \nu), u(x)) = 0 \); where the stress \( \sigma \) is function of material parameters \( E, \nu \) and displacement \( u(x) \). Given a measured displacement field, a piecewise-constant material parameter distribution is assumed and the distribution is evaluated through a finite element formulation.

This idea of equilibrium gap formulation can be written specifically for beams where the equilibrium equations can be written as function of the generalised forces at each section:

\[
F_i + F_j = P \quad \text{and} \quad M_i + M_j = T
\]

where the index \( i, j \) denote the internal force to the left and right of a given section, respectively. Variables \( P \) and \( T \) denote the nodal shear forces and moment, respectively at the section under consideration. In contrast with the 2D equilibrium, the external loading is included in the equilibrium expression of beams. The equilibrium of the internal forces and external loads must be satisfied along the beam.

The equilibrium conditions are directly applied to a FE formulation of the Euler-Bernoulli beams. Neglecting the axial deformation and considering an isotropic damage definition, the force displacement equations at the element level of the Euler-Bernoulli beam are given by:

\[
kd = (1 - D) \frac{EI_0}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix} = (1 - D)k_0d = f
\]

where \((v, \theta)\) are the nodal degrees of freedoms (vertical deflection, rotation angle), and \((V, M)\) are the corresponding nodal forces (shear, moment). Variable \( D \in [0,1] \) is a scalar considered as an index indicating the damage (reduction of stiffness). The reference stiffness is defined as \( EI_0 \), and can vary across the beam, i.e. it stands for \( EI_0(x) \). A possible definition of the damage variable is, \( D = 1 - \frac{EI}{EI_0} \), where \( EI \) is the actual stiffness of the cross section. Using such definition, the beam’s stiffness is expressed as a function of the damage index \( D \), \((k = (1 - D)k_0)\), where \( k_0 \) is the stiffness matrix of an undamaged or reference element. For each element, a constant damage parameter \( D \) is considered, which corresponds to a piecewise-constant definition of damage field along a beam. The parameter \((1 - D)\) corresponds to the stiffness reduction factor.
We will assume that the deflection profile is available from measurements through digital images, for each pair of two adjacent elements \(i\) and \(j\), we have the following expression:

\[
(1 - D_i)(k_0^3)_{i}d_i + (1 - D_j)(k_0^3)_{j}d_j = (V_2)_{i} + (V_1)_{j} = P_{ij}
\]

\[
(1 - D_i)(k_0^3)_{i}d_i + (1 - D_j)(k_0^3)_{j}d_j = (M_2)_{i} + (M_1)_{j} = T_{ij}
\]

where \((k_0^3)_{i}\) is the third row in the reference stiffness matrix of element \(i\); and \(D_i\) is the damage index of element \(i\). Similarly, \((k_0^3)_{j}\) is the first row of the reference stiffness matrix of element \(j\). The generalised forces \(P_{ij}\) and \(T_{ij}\) are the concentrated load and moment applied at the node connecting element \(i\) and \(j\), respectively. Writing these equilibrium equations for each pair of adjacent elements, we have a system of equations:

\[
\mathbf{A} \mathbf{\theta} = \mathbf{R}
\]

\[
\mathbf{A} = \begin{bmatrix}
(k_0^3)_{1}d_1 & (k_0^3)_{2}d_2 \\
(k_0^4)_{1}d_1 & (k_0^4)_{2}d_2 \\
(k_0^4)_{2}d_2 & (k_0^4)_{3}d_3 \\
(k_0^4)_{2}d_2 & (k_0^4)_{3}d_3 \\
(k_0^4)_{3}d_3 & \cdots \\
(k_0^4)_{3}d_3 & \cdots \\
\vdots & \vdots \\
\end{bmatrix}
\]

\[
\mathbf{\theta} = (1 - D_1 \ 1 - D_2 \ \cdots \ 1 - D_{nel})^T, \quad \mathbf{R} = (P_1^T \ M_1^T \ \cdots \ P_i^T \ M_i^T \ \cdots)^T
\]

The system of linear equations (4) is over-determinate. With the unavoidable noise in measured displacements, the standard least-square solution can be unstable. A total-variation (TV) regularization scheme is used to solve this system [20].

On the practical side, the measurement of the rotation angles of beams is very difficult. Different techniques can be used to overcome this difficulty: (i) a static condensation can be used to eliminate the degrees of freedoms associated to the rotations, (ii) estimate the angles from measured displacements using a stable numerical differentiation such as mollification [21], or (ii) through a minimization of total strain energy while treating the angles as unknown variables.

The equilibrium conditions at the supports or end points are excluded; thus this formulation can be applied for both statically determinate and indeterminate beams. Moreover, the concentrated loads applied to the beam need to be excluded in writing the equations (3) and (4); therefore the regions close to the loads must be excluded from the identification.
DATA DISCEPANCY BASED FORMULATION

Having measurements of the beam’s deflection, the reconstruction of the stiffness of a beam can be formulated as:

$$\min_{\theta} J(u, \theta) = \frac{1}{2} \sum_{i=1}^{N} |u_i(\theta) - u^m_i|^2 + \alpha \Phi(\theta)$$

$$s.t. \quad K(\theta)u = f$$

(5)

where $N$ is the total number of measurements. $\theta$ is the vector of unknown parameters (the beam’s stiffness distribution in the present case). The vector $u^m$ is the measured data displacement and $u(\theta)$ the corresponding simulated corresponding to a set of distribution parameters $\theta$. The constraint appearing the minimization problem (5) is the solution by finite element of a direct problem corresponding to a set of parameters $\theta$.

The cost function contains the sum of squared differences between measured and simulated deflection profile of the beam and a regularization term. Regularization is necessary to stabilize the numerical solution, and to ensure uniqueness. Similar to the equilibrium gap method, the total-variation (TV) regularization is also used here; numerical simulations showed that TV regularization gives better results than the classical Tikhonov method. However, the implementation of TV-regularization is different when dealing with linear system of equations or an optimization problem; implementation details can be found in [20][22].

To solve the minimisation problem, we need to evaluate the gradient of the functional. The adjoint method is an efficient algorithm that allows computing the gradient of a cost functional. It is employed in the present work to reconstructs the stiffness distribution through the solution of the constrained optimization problem given in (5). The Adjoint method is used here to compute the gradient vector of the objective function to the unknown stiffness distribution parameters.

The constraint equations are the equilibrium governing equations of the discretized beam using finite element; they are written in the following form:

$$R(\theta, u(\theta)) = K(\theta)u - f = 0$$

(6)

where $u$ is the state variables (displacements in the present case), which is an implicit function of the unknown material variables $\theta$. We introduce Lagrangian multipliers to modify the constrained optimization problem to an unconstrained optimization. The augmented functional that enforces the governing equations via Lagrange multipliers is expressed as:

$$L(\theta, u) = J(\theta, u) - \lambda^T R(\theta, u)$$

(7)
Differentiating the Lagrangian with respect to $\theta_i$, the gradient of the Lagrangian is given by:

$$
\frac{dL(\theta, u)}{d\theta_i} = \frac{\partial J}{\partial \theta_i} + \left( \frac{\partial J}{\partial u} - \lambda^T \frac{\partial R}{\partial u} \right) \frac{du}{d\theta_i} + \lambda^T \frac{\partial R}{\partial \theta_i}
$$

(8)

In general, the term $du/d\theta_i$, the derivative of displacement to the material parameters, is difficult to evaluate. By observing that by a suitable choice of $\lambda$, it is possible to make the term in the bracket equals zero, then avoid the need to evaluate the gradient $du/d\theta_i$. A new equation can be written:

$$
\left( \frac{\partial J}{\partial u} - \lambda^T \frac{\partial R}{\partial u} \right) = 0
$$

(9)

Equation-(9) is the adjoint equation; we solve this equation for $\lambda^T$. In this particular case, equation (9) can be expressed as:

$$(U - U^m) = \lambda^T K(\theta)$$

(10)

where $U$ is the displacement vector computed from the finite element equation-(6), and $U^m$ is the measured displacement vector. Therefore we have:

$$
\frac{dL(\theta, u)}{d\theta_i} = \frac{\partial J}{\partial \theta_i} + \lambda^T \frac{\partial R}{\partial \theta_i}
$$

(11)

in which the two derivatives can be easily computed:

$$
\frac{\partial J}{\partial \theta} = \alpha \frac{\partial \Phi}{\partial \theta}
$$

(12)

$$
\frac{\partial R}{\partial \theta_i} = \frac{\partial K(\theta)}{\partial \theta_i} - \frac{\partial f}{\partial \theta_i}
$$

(13)

The term $\Phi$ in (12) is the TV-regularization function; the detailed expression for its gradient to $\theta$ can be found in [22]. The global stiffness matrix is the assembly of the elements’ stiffness matrices; and for linear elastic materials, the element stiffness matrix is function of the material elastic properties and the geometric characteristics of the cross-section. Therefore, the derivative of the stiffness matrix to $\theta$, which is either the damage indices, or the stiffness reduction factors as defined in the previous section, and it can be computed analytically.

In conclusion, the adjoint method consists of two consecutive steps: 1) solve equation-(9) for $\lambda^T$, then insert $\lambda^T$ into equation-(11) to calculate the gradient of Lagrangian. With the gradient calculated from adjoint method, classical gradient-based optimization techniques can be used to find the unknown stiffness parameters. In the present paper the gradient-
assisted optimization algorithm available in MATLAB optimization toolbox is used in the simulation illustrated below [23].

**NUMERICAL EXAMPLES**

**Application of the Equilibrium gap method**

A 10m long simply-supported beam is simulated to demonstrate this method. This beam is assumed to have a constant stiffness along its span; a concentrated load of 5 kN is applied at its center. The stiffness at 2.5 m and 6 m from the left support are reduced to 50% and 70% of the original value (D = 0.5 and D=0.7), respectively. The simply-supported beam is discretized using 100 beam elements; hence, the continuous stiffness distribution function is represented by 100 discrete unknown parameters. A finite element simulation of the deflection profile is used as the measured value. A Gaussian noise is added to the simulated response to emulate real measured signals as expressed by equation (14).

\[
\text{noise}(t) = NRND \cdot a \cdot \text{RMS}(u)
\]  

(14)

where NRND is a Gaussian (normal) random number with zero-mean and unit standard deviation; a is the applied noise level; and RMS(u) is the root-mean-square of the measured displacement u(x). In the numerical examples angles are obtained from displacements by minimization of strain energy. The TV-regularized equations are solved using a Primal-dual method detailed in [22].

Figure 1-(a) shows the identified stiffness with no noise added to the deflection measurements; the two damaged zones are perfectly identified; the small variations of damage index are due to the error induced by the recovery of rotation angles from displacements. In Figure 1-(b) and (c), noise levels of 2.5% and 5% are added to the measured deflection data, respectively. The numerical simulations still show the position of the damage. However, it is observed that the noise in the measurement can mask fine-scale damages. If the level of noise is known, the minimum scale of detectable damage corresponding to this level of noise can be obtained by visual inspection. For example, at a 5% noise level, the finest detectable level of damage seems to be 10%. In Figure 1-(d), 10% noise is added, and the damages levels estimated are 30% and 10% for the two regions.
Application of the Adjoint optimization method

In this example, the problem consists at reconstructing the stiffness of a beam from the measurement of the deflection curve. For this validation, the experimental data are emulated from a finite element analysis. A noise level of 5% is introduced to the emulated displacement field. For the first test, we try to recover the stiffness of a beam with constant stiffness. The initial guess takes a uniform stiffness factor of 0.5. The identified and reference distribution of stiffness factors are shown in Figure 2. The results show a good performance of the proposed procedure for regions away from the supports. The displacement of a simply supported beam close to the support are very small (zero at the support), there is not enough information available in this region to improve the initial guess.

For the second case, we attempt to reconstruct a beam’s stiffness with parabolic distribution. The results are shown in Figure 3. Again a constant stiffness factor of 0.5 is used for the initial guess to start the iteration.
CONCLUSION

Two novel inverse computational techniques, equilibrium gap method and adjoint-optimization method, are proposed and implemented for the identification of stiffness distribution and detection of damages in beams. The former is based on the equilibrium gap of the beam; and the latter uses the data discrepancy functional between measured and simulated displacements. Optical measurement of continuous deflection profile under static loading is used as input in these techniques. Two examples are illustrated to validate the accuracy and efficiency of these methods. The equilibrium gap method is efficient in identifying localized damage and the data discrepancy method is a good candidate for the recovery of continuous stiffness distribution. Experimental tests are to be conducted to study the performance of the two techniques in practical engineering applications.

REFERENCES


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