Evaluation of Failure Parameters in Laminates by means of Hierarchical Plate Models

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Keywords

Failure Index, Multilayered Structures, Principle of Virtual Displacements (PVD), Reissenr's Mixed Variational Theorem (RMVT, Equivalent Single Layer (ESL), Layer Wise (LW), C_z^0 Requirements, Tsai Wu's failure criterion, Von Mises's failure criterion.

Abstract

Assuming a Hierarchical Plate Modeling the analytical computation of failure parameters, such as failure indices and loads, according to the stresses based Von Mises and Tsai-Wu failure criteria is here addressed. Analysis have been performed considering the ratio between the plate side and thickness a/h as parameter in order to investigate the applicability of several plate models. For high values of a/h classical plate theories either present a good agreement or conservative results with respect to higher order models; instead for small values the adoption of higher order theories is required.

1 Introduction

The capability to predict failure in laminated structures is a keypoint for their rational design. For that scope several failure criteria have been developed. This work takes place from the preliminary consideration that behind the assumption of every failure criterion a crucial aspect is the possibility to adopt a model that is able to furnish as accurate as wanted stresses and strains fields. In that sense a plate models unifying and hierarchically classifying environment has been assumed (section 2). Under that work leitmotiv Von Mises and Tsai-Wu's failure criteria have been considered (section 3) and three analyses of plates structures are addressed (4). In those analyses attention is focused on the computation and comparison of failure parameters, such as minimum failure load and failure index, obtained via several unified plate theories.

2 Hierarchical Plate Models

The term plate means a structure whitout curvature whose two dimensions are dominant with respect to the third one perpendicular to them. The negligible dimension is the structure thickness and it identifies the z direction. The predominant dimension represent the plate sides and they characterize the in plane x and y directions and a reference plane Ω.

An unifying theoretical environment, [1, 2], is here addressed where a broad variety of plate models can be compared and, thus, hierarchically classified taking into account the following points:

1. polynomial expansion order along the thickness;
2. variational statement;
3. laminate or lamina description level.

2.1 Polynomial Expansion

Assuming an axiomatic leitmotiv, the dependence of every problem unknown component $f\left(x,y,z\right)$ with respect to the along the thickness coordinate z is opportunely postulated and the problem is reconducted to an in-plane behavior determination:

$$f\left(x,y,z\right)=\sum_{j=1}^{N}f_j\left(x,y\right)F_j\left(z\right) \quad (1)$$

being $F_j\left(z\right)$ the up to the N-order known polynomial functions of z and $f_j\left(x,y\right)$ the new two dimensional problem unknowns.

2.2 Variational Statements

A plate model can be characterize by means of the choice of the problem main unknowns or, equivalently, by the assumption of the variational statement. The hierachical plate models herein addressed are based either on the Principle of Virtual Displacements (PVD) or on the Reissenr's Mixed Variational Theorem (RMVT). A theory will straightly furnish the displacement field $u\left(x,y,z\right),\ v\left(x,y,z\right),\ w\left(x,y,z\right)$, if PVD is adopted:

$$\int_V\left(\delta\left\{\epsilon_{pG}\right\}^T\left\{\sigma_{pC}\right\}+\delta\left\{\epsilon_{nG}\right\}^T\left\{\sigma_{nC}\right\}\right)\,dV=\delta L_e \quad (2)$$

Subscripts p and n represent the in plane and out of plane stress/strain components respectively; subscript G means that strain components have been obtained by means of derivation of the displacements field and C stands for stress components computed via

the material constitutive equations (Hook's generalized law).
Plate models based on RMVT:

$$\int_V \left(\delta \left\{\epsilon_{pG}\right\}^T \left\{\sigma_{pC}\right\} + \delta \left\{\epsilon_{nG}\right\}^T \left\{\sigma_{nM}\right\} + \delta \left\{\sigma_{nM}\right\}^T \left(\left\{\epsilon_{nG}\right\} - \left\{\epsilon_{nC}\right\}\right) \right) dV = \delta L_e \quad (3)$$

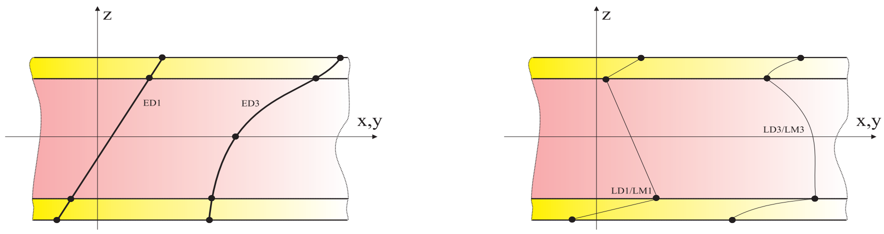
are characterized by a mixed solution nature, that is- either displacemts field and out of plane stress components $(\sigma_{zz}, \sigma_{xz}, \sigma_{yz})$ represent the main unknowns. Subscripts p, n, G, C in eq.(3) have the same meaning assumed in eq.(2), while subscript M indicates stress components coming from the plate model. In such a way it is possible, as it will be better explained further, to model a behavior due to transversal anisotropy and typical of laminates.

2.3 Description Level

In laminate structures attention can be turned toward the unknown quantities description level. Under a global point of view every unknown component is defined continuously above the whole laminate as expressed by means of eq.(1). This approach is known under the name of Equivalent Single Layer (ESL), as shown in figure 1.a. On the other hand, when a individual layer response is needed or in the case of significant unknowns gradients due to the presence of local phenomena such as localized loads that approach is no more suitable. In those cases a Layer Wise, that is- layer by layer, description has to be taken into account, see to figure 1.b. Eq.(1) in this context becomes:

$$f^k(x, y, z) = \sum_{j=1}^{N} f_j^k(x, y)\, F_j^k(z) \quad k = 1, \ldots, N_l \quad (4)$$

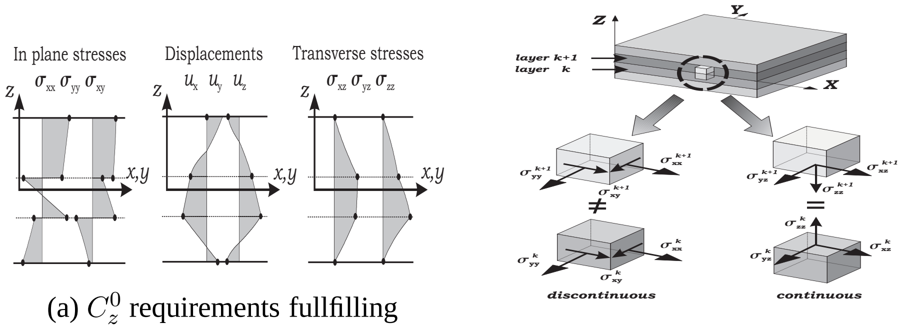
being N_l the total layers number. This approach requires the assumption of additional



(a) ESL plate model. (b) LW plate model.

Figure 1: Along the thickness solutions via ESL and LW.

conditions known as contact conditions between two adjacent layers. In this way it is possible to satisfy the so called C_z^0 requirements due to the transverse anisotropy and congruence/equilibrium conditions (as shown in figure 2.b): displacements and out of plane stress components must be continuous along the laminate thickness (Interlaminar continuity) and they must exhibit a change of slope at every layer interface (Zig-Zag Effect) as shown in figure 2.a. The assumption of a function that a priori bestows a zig-zag nature inside an ESL framework represents an attempt to model the zig-zag effect. An example of Zig-Zag function is the Murakami's one $(-1)^k \zeta_k$, where k profides the change of slope in correspondence of every layer and $\zeta_k = \dfrac{2z_k}{h_k}$ is a local layer dimensionless coordinates



(a) C_z^0 requirements fullfilling displacements stresses fields.

(b) congruence/equilibrium conditions.

Figure 2: C_z^0 requirements.

along the thickness. In this way eq.(1) becomes:

$$f(x,y,z) = \sum_{j=1}^{N-1} f_j(x,y)\,F_j(z) + (-1)^k \zeta_k f_N(x,y) \quad k = 1,\ldots,N_l \tag{5}$$

2.4 Acronyms System

At this point the definition of a way to hierarchically order the plate models with respect to all of the above addressed characteristics is needed. Figure 3 shows an acronyms system able to identify the plate models. The first letter (either L or E) specifies whether

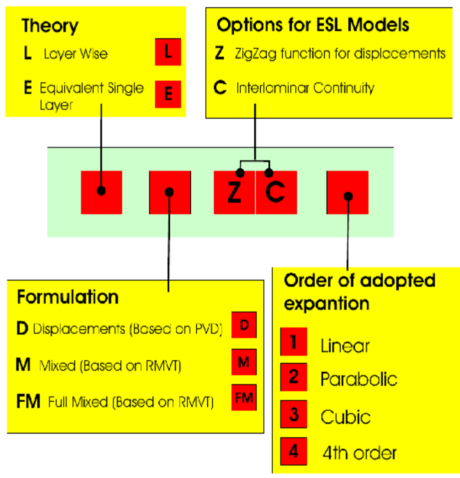
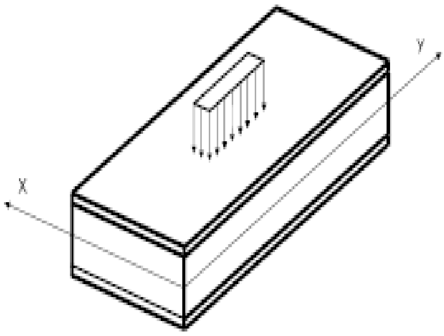


Figure 3: Hierarchical Plate Models acronyms.

a LW or a ESL approach has been assumed. The second letter indicates the adopted variational statement. The number at the end of the acronym represents the z coordinate expansion order. For ESL models also two additional letter can be used in order to specify the assumption of a zig-zag function and the fulfillment of the interlaminar continuity requirements.

2.5 Models Assessment

In order to demonstrate the validity and potentiality of all the addressed plate models environment the Meyer-Peining problem, [3], is taken into account. The structure is a sandwich plate whose geometrical data are provided in figure 4, while laminate layout and materials mechanical properties in table 1. Table 2 represents a comparison of the



Problem data:

- a = 100 mm;
- b = 200 mm;
- h_{top} = 0.1 mm;
- h_{bottom} = 0.5 mm;
- h_{total} = 12 mm;
- load value: 1 MPa;
- load y application extention: 20 mm;
- load x application extention: 5 mm;
- simply supported plate.

Figure 4: Meyer Piening's problem.

Laminate layout and materials mechanical properties									
Layer	E_{xx} [MPa]	E_{yy} [MPa]	E_{zz} [MPa]	E_{xz} [MPa]	E_{yz} [MPa]	E_{xy} [MPa]	ν_{xz}	ν_{yz}	ν_{xy}
1/3	$7.0\ 10^4$	$7.1\ 10^4$	$6.9\ 10^4$	$2.6\ 10^4$	$2.6\ 10^4$	$2.6\ 10^4$	0.3	0.3	0.3
2	3	3	2.8	1	1	1	0.25	0.25	0.25

Table 1: Meier-Piening's problem: materials mechanical properties.

in-plane normal stress σ_{xx} computed at plate center point with respect to the 3D solution obtained integrating the equilibrium equations [4]. Values obtained by means of the LM2 model are practically coincident with those computed via the exact 3D solution.

	z	top layer	$\|Err\%\|$	bottom layer	$\|Err\%\|$
3D	top	-624.00	-	-138.00	-
	bot	580.00	-	146.00	-
LM2	top	-619.49	0.72	-138.11	0.08
	bot	577.36	0.46	145.88	0.08
EMZC3	top	-486.07	22.10	-233.34	69.09
	bot	455.55	21.46	220.58	51.08
ED1	top	-30.02	95.19	4.38	103.17
	bot	-29.72	105.12	5.72	96.08

Table 2: Meier-Piening's problem: stress $\sigma_{xx}\,(a/2, b/2)$ at top and bottom of 1^{st} and 2^{nd} layer.

3 Failure Criteria

Two failure criteria has been taken into account. The first one is the Von Mises's failure criterion. It holds for isotropic elastic materials. In order to deal with orthotropic laminates Tsai Wu's failure criterion has been considered. For both of them a brief discussion follows. Attention is addressed toward the definition of the failure index and the failure load associated to an external pression $p_{zz}\,(x, y)$ loading.

3.1 Von Mises's Failure Criterion

According to this criterion, failure is due to the distortion strain energy associated to the deviatoric stress tensor. Failure occurs when that energy reaches the yielding level required in uniaxial loading. In such a way it is possible to define a stress level, σ_{eq}, representing a general three dimensional stress field:

$$\sigma_{eq} = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - \sigma_{xx}\sigma_{yy} - \sigma_{xx}\sigma_{zz} - \sigma_{yy}\sigma_{zz} + 3\left(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2\right)} \tag{6}$$

That equivalent stress has to be compared to the admissible yielding stress σ_y.
Failure index, Fvm, is the ratio of σ_{eq} and σ_y[1] and failure occurs when:

$$F_{vm} = \frac{\sigma_{eq}}{\sigma_y} \geq 1 \tag{7}$$

Once the stress field $\left\{\sigma_{ij}^{(0)} \quad i,j = x,y,z\right\}$ due to a generic pressure load $p_{zz}^{(0)}$ is known, thanks to the problem linearity it is possible to compute the stress state associated to any other value of the external pressure p_{zz}:

$$\frac{p_{zz}}{p_{zz}^{(0)}} = \frac{\sigma_{ij}}{\sigma_{ij}^{(0)}} \quad i,j = x,y,z \tag{8}$$

Considering eq.(6),and the failure condition, eq.(7), it is possible to obtain the expression of the failure load $p_{zz}^{(F)}$:

$$p_{zz}^{(F)} = \frac{p_{zz}^{(0)}}{\sigma_{eq}^{(0)}}\sigma_y \tag{9}$$

3.2 Tsai Wu's Failure Criterion

For this criterion the failure index F_{tw} is [5]:

$$F_{tw} = \sum_{i=1}^{3} F_i\sigma_i + \sum_{i=1}^{6} F_{ii}\sigma_i^2 + \sum_{i=1}^{2}\sum_{j=i+1}^{3} F_{ij}\sigma_i\sigma_j \tag{10}$$

where $(\sigma_1, \sigma_2, \sigma_3)$ are the normal stress components and $(\sigma_4, \sigma_5, \sigma_6)$ the shear stress $(23, 13, 12)$ components in principal material coordinates and:

$$\begin{gathered} F_1 = \frac{1}{X_T} - \frac{1}{X_C}, \quad F_2 = \frac{1}{Y_T} - \frac{1}{Y_C}, \quad F_3 = \frac{1}{Z_T} - \frac{1}{Z_C} \\ F_{11} = \frac{1}{X_T X_C}, \quad F_{22} = \frac{1}{Y_T Y_C}, \quad F_{33} = \frac{1}{Z_T Z_C} \\ F_{12} = -\frac{1}{2}\frac{1}{\sqrt{X_T X_C Y_T Y_C}}, \quad F_{13} = -\frac{1}{2}\frac{1}{\sqrt{X_T X_C Z_T Z_C}}, \quad F_{23} = -\frac{1}{2}\frac{1}{\sqrt{Y_T Y_C Z_T Z_C}} \end{gathered} \tag{11}$$

being (X_T, Y_T, Z_T) and (X_C, Y_C, Z_C) lamina normal strength in tension and compression respectively and (R, S, T) the lamina shear strength. Failure occurs when:

$$F_{tw} \geq 1 \tag{12}$$

Due to the problem linearity by imposition of the failure condition, eq.(12), it is possible to obtain the pressure failure load $p_{zz}^{(F)}$:

$$p_{zz}^{(F)} = \frac{-b + \sqrt{b^2 + 4a}}{2a} \tag{13}$$

[1] This is not the unique way of defining the failure index. Also the difference between σ_{eq} and σ_y, for example, could be adopted.

where:

$$a = \frac{1}{\left(p_{zz}^{(0)}\right)^2} \left[\sum_{i=1}^{6} F_{ii} \left(\sigma_i^{(0)}\right)^2 + \sum_{i=1}^{2} \sum_{j=i+1}^{3} F_{ij} \sigma_i^{(0)} \sigma_j^{(0)} \right]$$
$$b = \frac{1}{p_{zz}^{(0)}} \sum_{i=1}^{3} F_i \sigma_i^{(0)} \tag{14}$$

4 Case study

In these section for several plates the computation of the failure loads and indeces are presented. The ration a/h has been taken as analysis parameter in order to establish a models hierarchy.

4.1 Plate made of an isotropic layer subjected to an uniform load

An isotropic layer is here taken into account. Plate is subjected to an uniformly distributed load p$_{zz}$ = 0.1 N/mm^2 acting on its top. The sides have the same length equal to 240 mm. The material is the aluminum alloy 7075-T651: Young' modulus E is 71700 N/mm^2; Poisson's ration ν has the value 0.3; σ_y = 503 N/mm^2. For a/h the following values have been considered: 100, 50, 10, 5, 3. Table 3 shows values and locations of the maximum value of σ_{eq}, proportional to the failure index F_{vm}. First Order Shear Deformation Theory (FSDT) and ESL based on PVD models with several expansion order have been considered. Results obtained with FEM analyses via the Ansys FEM code are, also, presented. For $100 \leq a/h \leq 10$ there is a good agreement of values, while becoming the plate more and more thicker, $a/h = 5,\ 3$ there is a difference up to 25% in the results and FSDT does not provide conservative values, in such cases, thus, a higher order theory is required. The migration of the maximum value location from plate center to the corners increasing a/h it is explained by the fact that the value of σ_{zz} increases. It becomes of the same magnitudo order of σ_{xx} and σ_{yy} and the negative terms in eq.(6) assumes more relevance. In the corners this contribute is not present since $\sigma_{eq} = \sqrt{3}\tau_{xy}$. In table 4 minima

Maximum Von Mises's stress σ_{eq} $[N/mm^2]$					
a/h	100	50	10	5	3
ANSYS^1	322.899*	80.726*	3.229*	0.807*	0.291*
ANSYS^2	319.698*	79.929*	3.200*	0.800*	0.288*
ANSYS^3	312.407*	78.107*	3.128*	0.783*	0.282*
FSDT	329.860†	82.464†	3.299†	0.825†	0.297†
ED1	323.864†	82.472†	3.307†	0.833‡	0.305‡
ED2	323.069$^\diamond$	80.832$^\diamond$	3.286$^\diamond$	0.848$^\diamond$	0.319$^\diamond$
ED3	323.147$^\diamond$	80.882$^\diamond$	3.349*	0.937*	0.421*
ED4	323.148$^\diamond$	80.884$^\diamond$	3.338*	0.916*	0.396*

the following superscripts indicate the mesh elements number for side:
1: 120;
2: 24;
3: 12;

the following superscripts indicate the values location:
$\star$: center and corner points top and bottom;
$\dagger$: top and bottom of center point;
$\ddagger$: top of center point;
$\diamond$: top and bottom of corners points;
$*$: top of corners points.

Table 3: Maximum Von Mises's stress σ_{eq} via hierarchical plate theories and by Ansys.

failure load are shown. These values are strictly correlated to them shown in table 3 and the same considerations hold.

Minimum failure load $[N/mm^2]$					
a/h	100	50	10	5	3
FSDT	0.153	0.610	15.249	60.996	169.431
ED1	0.152	0.610	15.211	60.392	164.855
ED2	0.156	0.622	15.305	59.312	157.854
ED3	0.156	0.622	15.017	53.688	119.359
ED4	0.156	0.622	15.070	54.917	126.915

minima locations are the same to them shown in table 3

Table 4: Uniform load: mininum failure load via hierarchical plate theories.

4.2 Plate made of three isotropic layers subjected to a bi-sinusoidal load

A three layers laminate made of isotropic materials is here considered. The laminate configuration and the material properties are presented in table 5. Plate sides are equal to 300 mm and analyses have been performed considering $a/h = 100,\ 50,\ 10,\ 5,\ 3$. A bi-sinusoidal load of amplitude 0.1 N/mm^2 is applied on the plate top. In table 6 the minimum

Laminate configuration and material mechanical properties				
Layer	Material	$E\ [N/mm^2]$	ν	$\sigma_{eq}\ [N/mm^2]$
bottom	aluminum alloy 7075-T651	71700	0.33	503
center	titanium alloy Ti-6Al-4V	113800	0.33	950
top	AISI grade 18N maraging steel	183000	0.33	965

Table 5: Laminate layout and material properties.

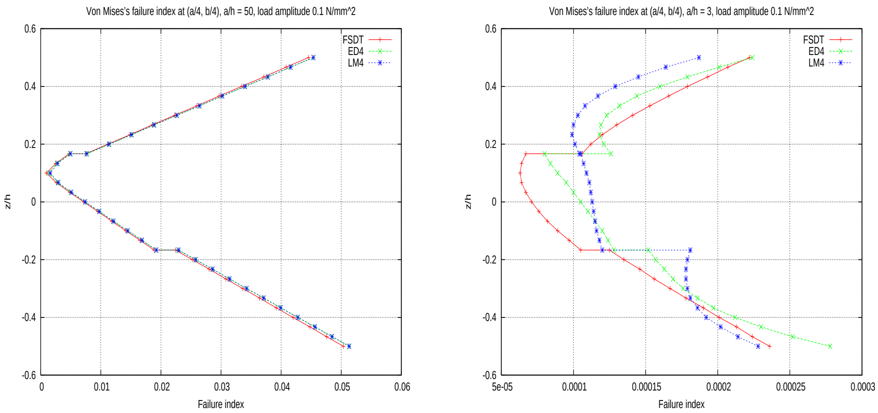
failure load is reported computed by means of several plate models. For $100 \leq a/h \leq 10$ all of the higher order theories substantially agrees, while FSDT provides a conservative value; for $a/h = 5$ the LD4 and LM4 provides concording results, while for $a/h = 3$ a highly accurate model such LM4 is required. Figures 5.a and 5.b show the failure index

Minimum failure load† $[N/mm^2]$					
a/h	100	50	10	5	3
FSDT	0.288	1.152	28.809	115.235	259.278
ED4	0.323	1.292	31.153	112.662	221.889
EDZ4	0.323	1.292	31.659	123.840	252.730
LD4	0.323	1.293	32.287	128.872	276.409
EM4	0.323	1.294	32.733	134.313	271.415
EMZ4	0.323	1.294	32.138	121.052	234.216
LM4	0.323	1.293	32.287	128.873	290.524

† all the maxima are at plate bottom center point.

Table 6: Three isotropic layers laminate: minimum failure load values and location.

plot along the thickness at $(a/4, b/4)$ for $a/h = 50$ and $a/h = 3$ respectively. In the first case the good agreement among all of the theories is underlined. In the second figure the out-of-plane stress components are predominant as it can be understood from figure 5.b at $z/h = 0.1$ where failure index is continuous at layers interfaces. This can be explained by the fact that the out-of-plane stress components must be of class C^0 for the interlaminar continuity and σ_y for those layers is practically equal (see table 5).



(a) $a/h = 50$ (b) $a/h = 3$

Figure 5: Von Mises's failure index along z via FSDT, ED4, LM4 in $(a/4,\ b/4)$, load amplitude $0.1\ N/mm^2$.

4.3 Plate made of three orthotropic layers subjected to a bi-sinusoidal load

In this case a laminate made of T300/5208 graphite/epoxy is analyzed. The lamination is $90/0/90$ and material properties are listed in table 8. Plate sides measure 300 mm each and $a/h = 100,\ 50,\ 5$ has been considered. A bi-sinusoidal load of amplitude 0.1 N/mm^2 has been applied on plate top. Table 7 shows the minimum failure loads. For all of the models the minimum is locate at plate center point; the position along z is at the top of the second layer except for points marked by † and $\star$. For $a/h = 100$ and 50 higher order models are in good agreement, in those cases FSDT is conservative but it underestimates too much the result. For $a/h = 5$ only LD4 and LM4 can be considered reliable. Figures 6.a and 6.b represent the behavior of the failure index along z at plate

Minimum failure load $[N/mm^2]$			
a/h	100	50	5
FSDT	0.003	0.011	1.318
ED4	0.025	0.101	3.380
EDZ4	0.028	0.115	6.753
LD4	0.025	$0.114^{\dagger}$	$1.518^{\star}$
LM4	0.025	$0.111^{\dagger}$	$1.540^{\star}$

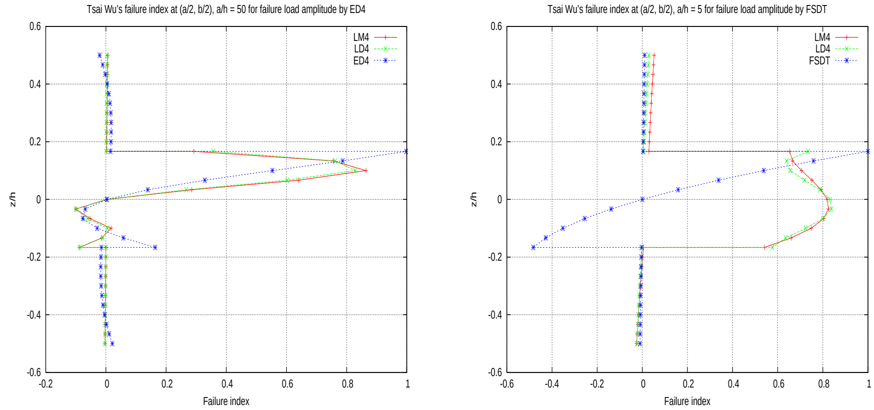
† Minimum at z = 0.6 mm
$\star$ Minimum at $z = -2$ mm

Table 7: Three orthotropic layers laminate: minimum failure load.

E_1 = 132379 N/mm^2;
E_2 = E_3 = 10756 N/mm^2;
G_{23} = 3378 N/mm^2;
G_{12} = G_{13} = 5654 N/mm^2;
$\nu_{23} = 0.49$, $\nu_{12} = \nu_{13} = 0.24$;
X_T = 1513 N/mm^2;
Y_T = Z_T = 44 N/mm^2;
X_C = 1696 N/mm^2;
Y_C = Z_C = 164 N/mm^2;
R = 68 N/mm^2, S = T = 87 N/mm^2.

Table 8: Material mechanical properties.

center point using a load equal to failure load obtained via ED4 and FSDT respectively. The agreement between LD4 and LM4 can be noticed.



(a) $a/h = 50$, ED4, LD4, LM4, for failure load computed by means of ED4

(b) $a/h = 5$, FSDT, LD4, LM4, for failure load computed by means of FSDT

Figure 6: Tsai Wu's failure index along z in $(a/2,\ b/2)$

5 Conclusions

The application of Hierarchical Plate Models hare been addressed and adopted in order to compute the failure parameters (failure index and failure load) using Von Mises's and Tsai Wu's failure criteria.

In the case of plate made of isotropic materials for $100 \leq a/h \leq 10$ classical models and higher order theories provide concording results; for $a/h = 5,\ 3$ LW models have to be preferred: FSDT does not always provide conservatives results so its application is suggested only for high values of a/h.

For orthotropic materials LW models present a good agreement for the considered values of a/h. FSDT is conservative but heavily underestimates the results.

References

[1] Carrera E., (2002). Theories and Finite Elements for Multilayered Anisotropic, Composite Plate and Shell. *Archives of Computational Methods in Engineering*, Vol. 9, 2, 87-140.

[2] Carrera E., (2003). Theories and Finite Elements for Multilayered Plate and Shell: A Unified Compact Formulation with Numerical Assessment and Benchmarking. *Archives of Computational Methods in Engineering*, Vol. 10, 3, 215-296.

[3] Meyer-Piening H. R., (2000). Experiences with 'Exact' linear sandwich beam and plate analyses regarding bending, instability and frequency investigations. *Proceedings of the Fifth International Conference On Sandwich Constructions*, Zurich, Switzerland, September 5-7, Vol. I, 37-48.

[4] Carrera E., Ciuffreda A. (2005). A unified formulation to assess theories of multilayered plates for various bending problems. *Composite Structures*,69, 271-293.

[5] Tsai S. W., (1987). *Composites Design*. Think Composites, Ohio, USA.