Theoretical Extension and Experimental Verification of a Frequency-Domain Recursive Approach to Ultrasonic Waves in Multilayered Media

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Abstract. The frequency-domain Scott and Gordon virtual-interface model for the propagation of ultrasonic waves in multilayered media was extended analytically and verified experimentally. Recursive compact equations were developed and introduced that enable to calculate the frequency features of ultrasonic waves propagating in compartments composed of any arbitrary number of layers. The calculations are carried out through an iterative process.
The spectra of experimental ultrasonic signals, obtained through a Fourier Transform were compared to the frequency features calculated by the model and were found to be in good agreement.

1. Introduction

In this work a model is developed in the frequency domain for the propagation of ultrasonic waves in multilayered compartments of an arbitrary number of layers. The motivation to develop the model in the frequency domain is that in many applications the frequency response and features are those of interest, and that some mathematical operations are more convenient in the frequency domain. As an example, consider the convolution integral in the time domain, which becomes a simple multiplication in the frequency domain.

This work extends analytically and explicitly the Scott and Gordon model in the frequency domain [1], for any number of layers. Recursive equations were developed and introduced, that enable to calculate the frequency features of ultrasonic waves propagating in multilayered compartments composed of any arbitrary number of layers. The equations are compact, and the calculations are carried out through an iterative process.

To verify and validate the model, numerous experiments were carried out, in which many compartments composed of different layers were tested ultrasonically. The spectra of the experimental ultrasonic signals, obtained through a Fourier Transform were compared to the frequency features calculated by the model.

The experimental results and the theoretical simulations turned out to be in good agreement. The model has proven to accurately provide detailed features of the ultrasonic spectra of layered structures.

2. The Scott-Gordon Approach in the frequency domain

Unlike the fundamental analysis carried out by Brekhovskikh (2) in the time-domain, Scott and Gordon conducted their analysis in the frequency-domain.

Consider the planar geometry of Figure 1. Let waves incident from the left upon medium 2 have transmission and reflection coefficients T12 and R12, and let those incidents from the right have coefficients T21 and R21. The coefficients for the boundary between 2 and 3 are analogously defined [1].

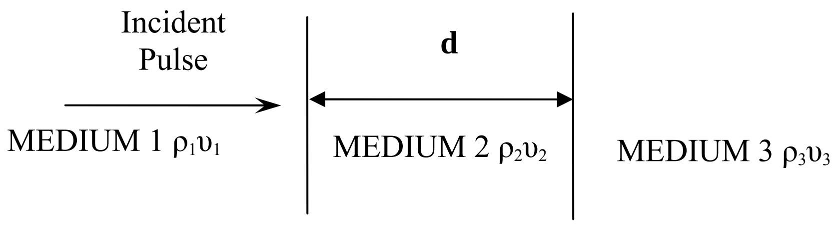

Fig.1. Ultrasonic pulse incident upon medium of finite thickness [1].

A portion of the pulse *A(t)* incident from the left onto medium 2 will be transmitted with amplitude $T_{12}A(t)$ and, upon emerging in medium 3, will produce a disturbance of the form

$$T_{12}T_{23}A(t-d/\upsilon).$$

In addition to the portion of the wave transmitted into medium 2, there will be a number of pulses emerging, which have undergone multiple reflections within this medium. The total of all such waves is given by

$$A'_{13}(t) \equiv T_{12}T_{23}A(t-d/\upsilon)+T_{12}R_{23}R_{21}T_{23}A(t-3d/\upsilon)\cdots, \qquad 1$$

$$A'_{13}(t) = T_{12}T_{23}\sum_{N=0}^{\infty}\left(R_{23}R_{21}\right)^{N} A(t-\tfrac{(2N+1)d}{\upsilon}), \qquad 2$$

where $\upsilon=\upsilon_2$. Similarly, the total wave reflected back into medium 1 is given by

$$R'_1(t) \equiv R_{12}A(t)+T_{12}T_{21}R_{21}^{-1}\sum_{N=1}^{\infty}\left(R_{23}R_{21}\right)^{N} A(t-2Nd/\upsilon). \qquad 3$$

Taking the complex Fourier transforms of expressions (2) and (3), we have:

$$A'_{13}(\omega) = \left[T_{12}T_{23}e^{-j\omega d/\upsilon}\sum_{N=0}^{\infty}\left(R_{23}R_{21}\right)^{N} e^{-2j\omega N d/\upsilon}\right]A(\omega), \qquad 4$$

$$R'_1(\omega) = \left[R_{12}+T_{12}T_{23}e^{-2j\omega d/\upsilon}\sum_{N=0}^{\infty}\left(R_{23}R_{21}\right)^{N} e^{-2j\omega N d/\upsilon}\right]A(\omega), \qquad 5$$

where $A'_{13}(\omega)$ denotes the Fourier transform of $A'_{13}(t)$ and $R'_1(\omega)$ is the Fourier transform of $R'_1(t)$, etc.

The sum appearing in expressions (2.21) and (2.22) can be readily evaluated for any finite or infinite N, as convergence geometric series, allowing the definition of generalized frequency-dependent reflection and transmission coefficients *r(ω)* and *t(ω)* for the layered medium. For N infinite, we have the expression

$$t_1(\omega) \equiv \frac{A'_{13}(\omega)}{A(\omega)} = \frac{T_{12}T_{23}e^{-j\omega d/\upsilon}}{1-R_{23}R_{21}e^{-2j\omega d/\upsilon}}, \qquad 6$$

$$r_1(\omega) \equiv \frac{R_1'(\omega)}{A(\omega)} = \frac{T_{12}T_{21}R_{23}e^{-2j\omega d/\upsilon}}{1-R_{23}R_{21}e^{-2j\omega d/\upsilon}} + R_{12}. \quad 7$$

Clearly, for the case of layered media of finite thickness a single reflection coefficient does not exist for the time-dependent representation of the wave A(t); however, we can define operators $\hat{R}$ and $\hat{T}$ which when operating on A(t) generate the expressions for the time-dependent reflected and transmitted waves, respectively [1].

The appropriate definition for these operations is

$$\hat{R}A(t) = \mathcal{F}^{-1}\left\{r(\omega)\mathcal{F}\left(A(t')\right)\right\}, \quad 8$$

$$\hat{T}A(t) = \mathcal{F}^{-1}\left\{t(\omega)\mathcal{F}\left(A(t')\right)\right\}, \quad 9$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and its inverse.

The use of this operator notation permits a layer to be represented as a boundary or virtual interface having these operators as their reflection and transmission coefficients.

2.1. Ultrasonic pulse incident upon layered media – the virtual layer approach

Consider the layered media in Fig.2. This is essentially the same as Fig.1.except that the boundary between media 2 and 3 is now replaced by a boundary medium with transmission and reflection operators $\hat{R}_{23}$ and $\hat{T}_{23}$.

Rewriting expressions (2.19) and (2.20) in the form of a series and replacing T_{23} and R_{23} by operators we have

$$A_{13}'(t) = T_{12}\hat{T}_{23}\sum_{N=0}^{\infty}\left(\hat{R}_{23}R_{21}\right)^N A\left(t-\tfrac{(2N+1)d}{\upsilon}\right), \quad 10$$

$$R_1'(t) \equiv R_{12}A(t) + T_{12}T_{21}\hat{R}_{23}\sum_{N=0}^{\infty}\left(\hat{R}_{23}R_{21}\right)^N A\left(t-\tfrac{(2N+2)d}{\upsilon}\right). \quad 11$$

Using basic transform pair properties and assuming appropriate Dirichlet principle and absolute convergence properties, it follows that

$$\hat{T}\hat{R}A(t) = \hat{R}\hat{T}A(t)$$

$$\mathcal{F}(R^N T^N A(t)) = r^N(\omega)t^N(\omega)\mathcal{F}(A(t)).$$

Applying these results to take the transform of expressions 10 and 11, we have

$$T'(\omega) = A_{13}'(\omega)/A(\omega) = T_{12}t_1(\omega)\sum_{N=0}^{\infty}\left[r_1(\omega)R_{21}\right]^N e^{-j\omega(2N+1)d/\upsilon}, \quad 12$$

$$R' = R_{12}(\omega) + T_{12}T_{21}r_1(\omega)\sum_{N=0}^{\infty}\left[r_1(\omega)R_{21}\right]^N e^{-j\omega(2N+2)d/\upsilon}. \quad 13$$

Summing the series as before, we have

$$T'(\omega) = \frac{T_{12}t_1(\omega)e^{-j\omega d/\upsilon}}{1-r_1(\omega)R_{21}e^{-2j\omega d/\upsilon}}, \quad 14$$

$$R'(\omega) = \frac{T_{12}T_{23}r_1(\omega)e^{-2j\omega d/\upsilon}}{1-r_1(\omega)R_{21}e^{-2j\omega d/\upsilon}} + R_{12}. \quad 15$$

Normal incidence reflection and transmission coefficients for several finite arrays (for medium of 1-5 layers) have been calculated using formulas 14,15 by W. R. Scott and P. F. Gordon in order to get graphical form of dependences of the coefficients [1].

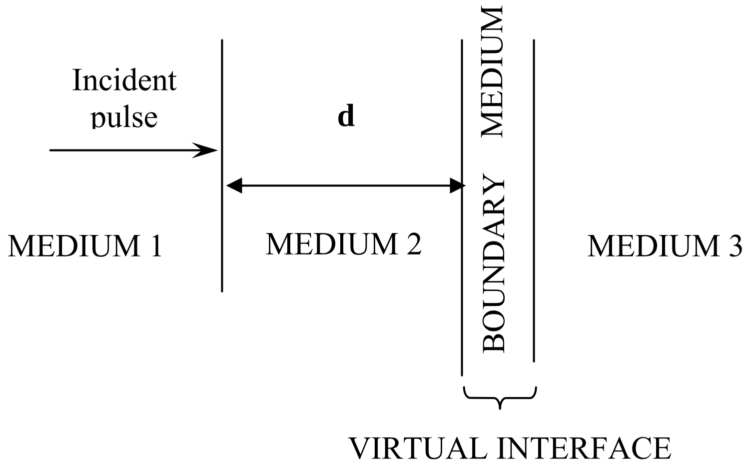

Figure 2: Ultrasonic pulse incident upon layered media [1].

2.2. Development of the Virtual Interface Model (VIM) in the frequency domain

2.2.1. Mathematical considerations and development of the VIM

Consider a multilayered compartment and the case of normal incidence. Additional mathematical calculations based on the equations listed above lead to the expressions of transmission and reflection coefficients for the first layer of the multilayered medium, when we count in reverse order:

$$t_{n+1,n}(\omega) = \frac{T_{1,n}T_{n,1}e^{-j\Phi_{n+1}}}{1+R_{1,n}R_{n,1}e^{-2j\Phi_{n+1}}}, \qquad 16$$

$$r_{n+1,n}(\omega) = \frac{R_{1,n}+R_{n,1}e^{-2j\Phi_{n+1}}}{1+R_{1,n}R_{n,1}e^{-2j\Phi_{n+1}}}, \qquad 17$$

Where

$$\Phi_i = \frac{\omega \cdot d_i}{\upsilon_i}, \qquad 18$$

n is a number of layers of the n-layered medium, when layers are enumerated from *i=2* to *n+1*, and *1* is an index of the water layer (above and under the n-layered medium).

The expression of transmission and reflection coefficients for the second layer of the multilayered medium can be written as

$$t_{n,n}(\omega) = \frac{T_{1,n}t_{n+1,n}(\omega)e^{-j\Phi_n}}{1+r_{n+1,n}(\omega)R_{1,n}e^{-2j\Phi_n}}, \qquad 19$$

$$r_{n,n}(\omega) = \frac{R_{1,n}+r_{n+1,n}(\omega)e^{-2j\Phi_n}}{1+r_{n+1,n}(\omega)R_{1,n}e^{-2j\Phi_n}}. \qquad 20$$

For the third layer of the multilayered medium we can write the expression of transmission and reflection coefficients in the form

$$t_{n-1,n}(\omega) = \frac{T_{1,n-1}t_{n,n}(\omega)e^{-j\Phi_{n-1}}}{1+r_{n,n}(\omega)R_{1,n-1}e^{-2j\Phi_{n-1}}}, \qquad 21$$

$$r_{n-1,n}(\omega)=\frac{R_{1,n-1}+r_{n,n}(\omega)e^{-2j\Phi_{n-1}}}{1+r_{n,n}(\omega)R_{1,n-1}e^{-2j\Phi_{n-1}}}, \quad 22$$

And so on.
Then the expression for the coefficients of the last layer can be obtained as

$$t_{2,n}(\omega)=\frac{T_{1,2}t_{3,n}(\omega)e^{-j\Phi_2}}{1+r_{3,n}(\omega)R_{1,2}e^{-2j\Phi_2}}, \quad 23$$

$$r_{2,n}(\omega)=\frac{R_{1,2}+r_{3,n}(\omega)e^{-2j\Phi_2}}{1+r_{3,n}(\omega)R_{1,2}e^{-2j\Phi_2}}. \quad 24$$

Successive application of these formulas yields a simple iterative procedure for tabulating transmission and reflection coefficients for media having an arbitrary number of layers of varying thickness and acoustic parameters.

2.2.2 The recursive equations of the VIM

Consider multilayered medium with arbitrary number of layers n. Now, we can easily express recursive equations of the VIM by means of the formulation developed above. Note again, that we count the layers backwards, i.e. in a reverse order.

For a single layer, when **n = 1**

$$t_{k-1,n}(\omega)=\frac{T_{k-1,k}T_{k,k-1}e^{-j\Phi_k}}{1+R_{k-1,k}R_{k,k-1}e^{-2j\Phi_k}} \quad 25$$

$$r_{k-1,n}(\omega)=\frac{R_{k-1,k}+R_{k,k-1}e^{-2j\Phi_k}}{1+R_{k-1,k}R_{k,k-1}e^{-2j\Phi_k}} \quad 26$$

Where k=n+1, i.e. k=2 is the index of the layer, and k-1 is the index of the water layer.
For any number of layers, when n > 1

$$t_{k,n}(\omega)=\frac{T_{1,k}t_{k-1,n}(\omega)e^{-j\Phi_k}}{1+r_{k-1,n}(\omega)R_{1,k}e^{-2j\Phi_k}} \quad 27$$

$$r_{k,n}(\omega)=\frac{R_{1,k}+r_{k-1,n}(\omega)e^{-2j\Phi_k}}{1+r_{k-1,n}(\omega)R_{1,k}e^{-2j\Phi_k}} \quad 28$$

Where k = 2… n+1 is the index number of the layer and $t_{1,n}(\omega)$, $r_{1,n}(\omega)$ are calculated by means of Equations 25 and 26.

2.2.3 The algorithm of the VIM

According to the recursive equations 25-28 we built an algorithm for arbitrary number of layers – *N* and programmed it by means of Matlab 6.1 [4]. For example, for a three-layered compartment:

The expression of transmission coefficient for the second layer (since the first layer is water) of the multilayered medium was written as

$$t_2(\omega)=\frac{T_{12}T_{23}e^{-j\omega d/\upsilon}}{1-R_{23}R_{21}e^{-2j\omega d/\upsilon}}, \quad 29$$

And the expression of reflection coefficient

$$r_2(\omega) = \frac{T_{12}T_{21}R_{23}e^{-2j\omega d/\upsilon}}{1 - R_{23}R_{21}e^{-2j\omega d/\upsilon}} + R_{12}\,. \qquad 30$$

By using:

$$R_{ik} = -R_{ki}\,, \qquad 31$$

$$T_{ik}T_{ki} - R_{ik}R_{ki} = 1\,. \qquad 32$$

Now we can obtain for the last layer (here we start calculation from the last layer i.e. from the third layer instead of the first), after all substitutions, as follows

$$t_4(\omega) = \frac{T_{14}T_{41}e^{-j\Phi_4}}{1 + R_{14}R_{41}e^{-2j\Phi_4}}\,, \qquad 33$$

$$r_4(\omega) = \frac{R_{14} + R_{41}e^{-2j\Phi_4}}{1 + R_{14}R_{41}e^{-2j\Phi_4}}\,. \qquad 34$$

For the sequent layer (number two), with the prior layer as a virtual interface, we can get the coefficients as:

$$t_{34}(\omega) = \frac{T_{13}t_4(\omega)e^{-j\Phi_3}}{1 + r_4(\omega)R_{13}e^{-2j\Phi_3}}\,, \qquad 35$$

$$r_{34}(\omega) = \frac{R_{13} + r_4(\omega)e^{-2j\Phi_3}}{1 + r_4(\omega)R_{13}e^{-2j\Phi_3}}\,. \qquad 36$$

And finally, for the last layer (number one), with the two prior layers as a virtual interface we can find the coefficients as:

$$t_{234}(\omega) = \frac{T_{12}t_{34}(\omega)e^{-j\Phi_2}}{1 + r_{34}(\omega)R_{12}e^{-2j\Phi_2}}\,, \qquad 37$$

$$r_{234}(\omega) = \frac{R_{12} + r_{34}(\omega)e^{-2j\Phi_2}}{1 + r_{34}(\omega)R_{12}e^{-2j\Phi_2}}\,. \qquad 38$$

Examples of simulations of the frequency response from several multilayered compartments are demonstrated below:

As an example, a simulation of the frequency response from a compartment composed of 3 layers: Aluminum – Steel – Aluminum is demonstrated in Figure. 3.

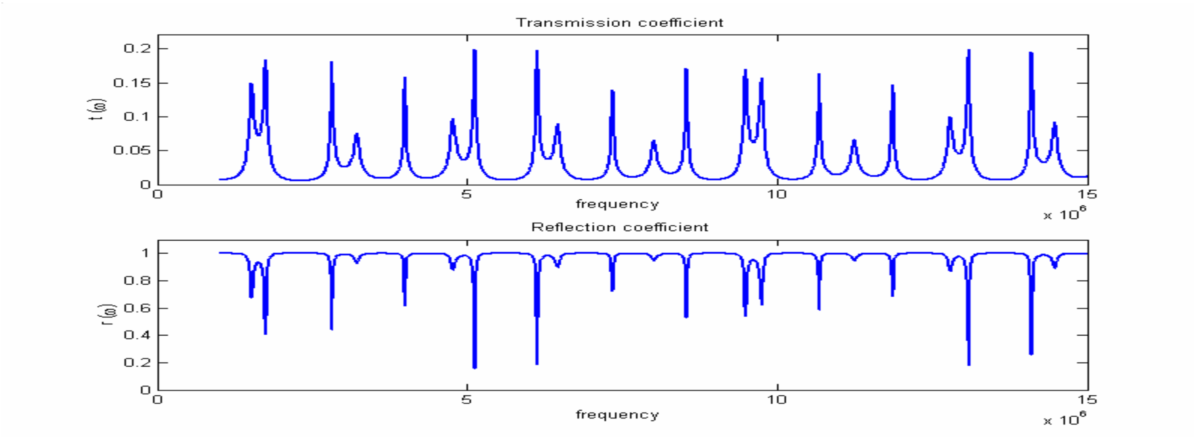

Fig. 3. The transmission and reflection coefficient of the 3-layered compartment (Aluminium – Steel – Aluminium).

3. Experiment

The goal of the experiment is to validate the Virtual Interface Model experimentally through ultrasound propagation in various multilayered media composed of various types of layers. The different compartments that were examined ultrasonically and compared to those calculated by means of VIM technique were: Al, 2*Al, 3*Al, 4*Al, St, 2*St, 3*St, 4*St, Al-St, Al-St-Al, Al-St-Al-St, Al-St-Al-St-Al, Al-St-Al-St-Al-St-Al, Al-St-Al-St-Al-St-Al-St, St-Al, St-Al-St, St-Al-St-Al-St-Al, St-Al-St-Al-St-Al-St-Al.

The experimental set-up consisted of regular and standard equipment including a computer, a digitizer, an immersion tank, a X-Y-Z position bridge, a pulser-receiver and an ultrasonic immersion focused probe of 5MHz.

The Figures below show Power Spectra of different compartments, composed of several layers of Aluminum, Steel and a mixture of both, obtained from the experiment and from simulations by means of the theoretical VIM technique. Consider some examples. A full description of the results is provided in [3].

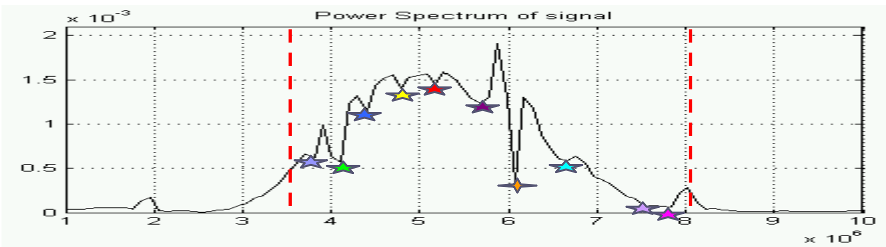

Figure. 4.A The experimental graph of power spectrum of the compartment composed of 4 plates of Steel and Aluminum: St-Al-St-Al

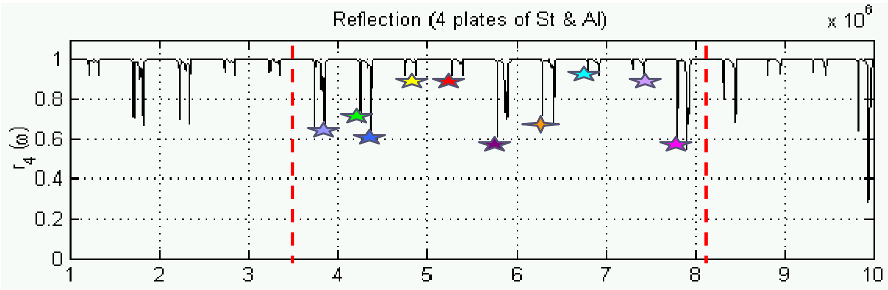

Figure. 4.B. The theoretical graph of power spectrum of the compartment composed of 4 plates of Steel and Aluminum: St-Al-St-Al

4. Summary and conclusions

The Virtual Interface Model (VIM) was developed to model the interactions of ultrasonic plane waves with laminated structures in the frequency domain. Using this model, measurable parameters such as the frequency-dependent transmission and reflection coefficients can be accurately calculated for laminated compartments consisting of layers of elastic monolithic materials.

We derived the recursive equations of the VIM for a compartment composed of an arbitrary number of layers (3<N) in the frequency domain. Successive application of these formulas yields a simple iterative procedure for calculating transmission and reflection coefficients for media having an arbitrary number of layers of varying thickness and acoustic parameters.

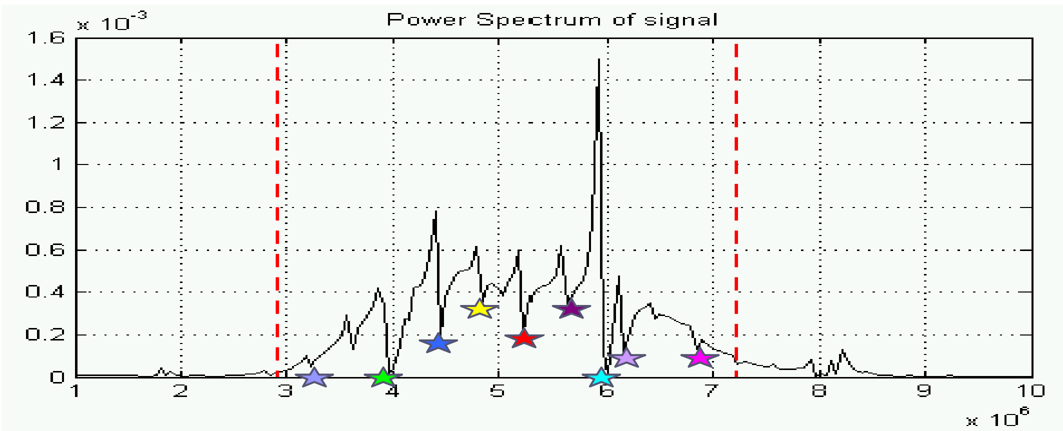

Figure 5.A: The experimental graph of power spectrum of the compartment composed of 8 plates of Aluminium and Steel: Al-St-Al-St-Al-St-Al-St.

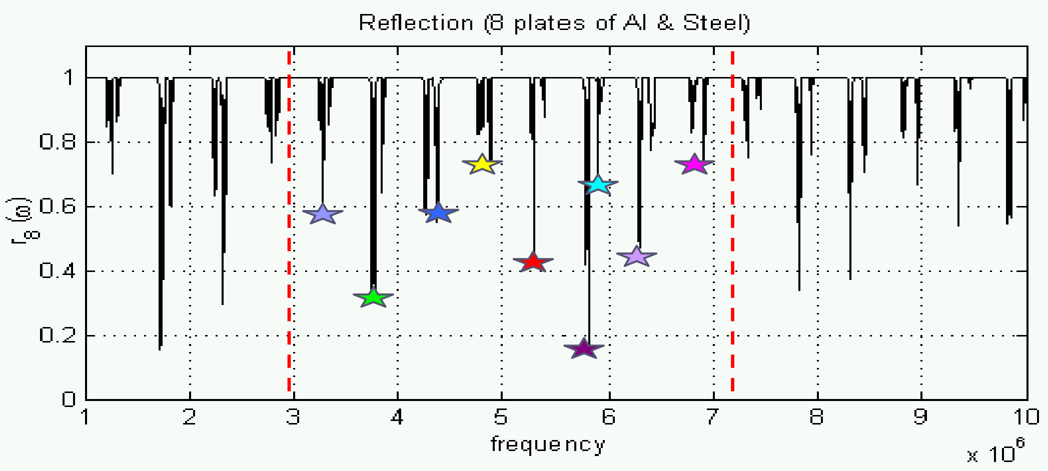

Figure 5.B.: The theoretical graph of power spectrum of the compartment composed of 8 plates of Aluminium and Steel: Al-St-Al-St-Al-St-Al-St.

By means of the recursive formulas we simulated the frequency response from different multilayered compartments, and in order to verify and validate the model we tested ultrasonically numerous compartments consisting of various types of layers. The spectra of the experimental ultrasonic signals, obtained through the Fourier Transform were compared to the frequency features calculated by means of the Virtual Interface model and turned out to be in good agreement.

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