Ultrasonic damping in polycrystalline materials
Anders Boström
Chalmers University of Technology, Dept. of Mechanics and Maritime Sciences, Gothenburg, Sweden, anders.bostrom@chalmers.se

Abstract
The grains in a polycrystalline material, typically metals, act as scatterers of ultrasonic waves and thus give rise to attenuation of the waves. The grains have anisotropic stiffness properties, typically orthotropic or cubic. A new approach is proposed to calculate damping in a 2D setting starting from the scattering by an anisotropic circle in an isotropic surrounding. This problem has recently been solved, giving explicit, simple expressions for the elements of the transition (T) matrix (which gives the relation between the incoming and scattered fields) when the circle is small compared to the ultrasonic wavelengths. The T matrix can be used to calculate the total scattering cross section, which in turn can be used to estimate the damping in the material. Explicit expressions for the damping coefficient for longitudinal and transverse waves are obtained in this way.

1. Introduction
The grains in a polycrystalline material, typically a metal, act as scatterers of ultrasonic waves and thus give rise to attenuation of the waves. To estimate the effective wave speed and attenuation in polycrystals various approximate methods have been used. As examples Stanke and Kino (1) calculate the wave speeds and attenuation using a perturbation method with weak anisotropy, Hirsekorn (2) performs a similar analysis for textured materials (giving an anisotropic effective material), Thompson et al. (3) give an overview of scattering of elastic waves in simple and complex polycrystals, and Li and Rokhlin (4) study the scattering in general random anisotropic solids. These studies all use volume integral equation methods combined with some perturbation method, in particular the Born approximation.

Here a different approach is proposed, where first the scattering by a single anisotropic grain in an isotropic surrounding is studied. Only the 2D problem for in-plane (P and SV) waves is treated and as the grains are assumed small compared to the wavelength it should be enough to assume the grain to be circular (the scattering at low frequencies is mostly a volume effect, see the above papers). The 2D P-SV scattering by an anisotropic circle in an isotropic surrounding has recently been solved by Boström (5). Then there exist many methods to calculate effective properties of a material with many inclusions, but here only the simplest approach to estimate the attenuation is investigated.
2. Scattering by an anisotropic circle

Consider the scattering of elastic P-SV waves by an anisotropic circle of radius $a$ residing in an isotropic medium. This problem has recently been solved by Boström (5) and here only the pertinent results are repeated. The material surrounding the circle has density $\rho_0$ and Lamé parameters $\lambda_0$ and $\mu_0$. Only time harmonic situations are considered and the time factor $\exp(-i\omega t)$, where $\omega$ is the angular frequency and $t$ is time, is suppressed throughout. The longitudinal (pressure, P) wave number is $k_p = \omega\sqrt{\rho_0/(\lambda_0 + 2\mu_0)}$ and the transverse (shear, S) is $k_s = \omega\sqrt{\rho_0/\mu_0}$.

A general way to represent the scattering by the circle is to calculate its T (transition) matrix, which relates the expansion coefficients of the scattered wave to those of the incoming wave. Close to the circle the incoming wave can be expanded in terms of regular wave functions

$$u^i = \sum_{\tau,\sigma,m} a_{\tau\sigma m} \chi^0_{\tau\sigma m},$$

where the summation is over $\tau = 1, 2$, $\sigma = e, o$, and $m = 0, 1, \ldots$, and $a_{\tau\sigma m}$ are the expansion coefficients that specifies the incoming wave. The scattered wave satisfies radiation conditions and can be expanded in terms of outgoing wave functions

$$u^s = \sum_{\tau,\sigma,m} f_{\tau\sigma m} \chi^+_{\tau\sigma m},$$

where $f_{\tau\sigma m}$ are the expansion coefficients of the scattered wave. For the definition of the 2D wave functions see the Boström (5) or (with a slightly different normalization) Varadan et al. (7). It is noted that the first index $\tau = 1$ gives the SV waves functions and the second $\tau = 2$ gives P wave functions. The relation between the scattered and incoming waves defines the transition matrix

$$f_{\tau\sigma m} = \sum_{\tau',\sigma',m'} T_{\tau\sigma m,\tau'\sigma'm'} a_{\tau'\sigma'm'}. \tag{3}$$

The transition matrix becomes symmetric with the present definition of the wave functions.

The material inside the circle is taken as orthotropic. It has density $\rho_1$ and stiffness constants $c_1$, $c_2$, $c_3$, and $c_4$, and the constitutive equations in Cartesian coordinates are thus

$$\sigma_{xx} = c_1 \epsilon_{xx} + c_3 \epsilon_{yy}, \tag{4}$$
$$\sigma_{yy} = c_3 \epsilon_{xx} + c_2 \epsilon_{yy}, \tag{5}$$
$$\sigma_{yx} = 2c_4 \epsilon_{xy}, \tag{6}$$

where $\sigma_{ij}$ and $\epsilon_{ij}$, $i, j = x, y$, are the stress and strain components, respectively, and the $x$ and $y$ directions are chosen along the principal directions. To be able to apply the boundary conditions at the circular boundary it is much more convenient to get solutions in polar coordinates and the constitutive equations are therefore transformed to polar coordinates

$$\sigma_{rr} = (\alpha_1 + 2\alpha_2) \epsilon_{rr} + \alpha_1 \epsilon_{r\varphi} + 2\beta_1 (\cos 2\varphi \epsilon_{rr} - \sin 2\varphi \epsilon_{r\varphi}) + \beta_2 (\cos 4\varphi (\epsilon_{rr} - \epsilon_{r\varphi}) - 2 \sin 4\varphi \epsilon_{r\varphi}), \tag{7}$$

2
\[ \sigma_{\varphi \varphi} = \alpha_1 \epsilon_{rr} + (\alpha_1 + 2\alpha_2)\epsilon_{\varphi \varphi} - 2\beta_1(\cos 2\varphi \epsilon_{\varphi \varphi} + \sin 2\varphi \epsilon_{r \varphi}) \\
+ \beta_2(\cos 4\varphi (\epsilon_{\varphi \varphi} - \epsilon_{rr}) + 2\sin 4\varphi \epsilon_{r \varphi}), \quad (8) \]

\[ \sigma_{r \varphi} = 2\alpha_2 \epsilon_{r \varphi} - \beta_1 \sin 2\varphi (\epsilon_{rr} + \epsilon_{\varphi \varphi}) + \beta_2 (\sin 4\varphi (\epsilon_{\varphi \varphi} - \epsilon_{rr}) - 2\cos 4\varphi \epsilon_{r \varphi}). \quad (9) \]

Here the following stiffness constants are introduced

\[ \alpha_1 = \frac{1}{8} (c_1 + c_2 + 6c_3 - 4c_4), \quad (10) \]

\[ \alpha_2 = \frac{1}{8} (c_1 + c_2 - 2c_3 + 4c_4), \quad (11) \]

\[ \beta_1 = \frac{1}{4} (c_1 - c_2), \quad (12) \]

\[ \beta_2 = \frac{1}{8} (c_1 + c_2 - 2c_3 - 4c_4). \quad (13) \]

It is noticed that for isotropic media these constants reduce to \( \alpha_1 = \lambda_1, \alpha_2 = \mu_1, \)
and \( \beta_1 = \beta_2 = 0, \) where \( \lambda_1 \) and \( \mu_1 \) are the Lamé constants of the circle. Thus \( \beta_1 \)
and \( \beta_2 \) are measures of the degree of anisotropy while \( \alpha_1 \) and \( \alpha_2 \) can be regarded as
some mean stiffnesses.

The displacement field inside the circle is divided into four independent parts
due to the symmetries. Thus the four parts are symmetric or antisymmetric with
respect to the \( x \) and \( y \) axes. Here a symmetric displacement with respect to the \( x \)
axis has an even \( x \) component and an odd \( y \) component and vice versa for symmetry
with respect to the \( y \) axis. The displacement components inside the circle can be
expanded in trigonometric series in the azimuthal coordinate appropriate to the
symmetry of the component. A power series in the radial coordinate is then assumed
and recursion relations can be set up for the expansion coefficients in these series,
essentially solving the problem inside the circle.

3. The low frequency T matrix

To determine the T matrix continuity of displacement and traction at \( r = a \) is set up
and the resulting systems are solved for the expansion coefficients of the scattered
field. As in the previous section the symmetries lead to a decoupling into parts that
are symmetric or antisymmetric with respect to the coordinate axes, thus leading
to a decomposition of the T matrix into four parts in the following way

\[ T_{\tau m, \tau' m'} = T^{j j'}_{\tau m, \tau' m'}. \quad (14) \]

where \( j = s \) or \( j = a \) and likewise for \( j' \), where \( s \) stands for symmetric and \( a \) for
antisymmetric and the first index \( j \) stands for symmetry with respect to the \( x \) axis
and the second \( j' \) for symmetry with respect to the \( y \) axis.

For the doubly symmetric case only even \( m \) values appear in the expansions and
for low frequencies only \( m = 0 \) and \( m = 2 \) contribute. To leading order the monopole
element for \( m = m' = 0 \) becomes

\[ T_{20, 20}^{ss} = (k_p a)^2 \frac{\pi \mathbb{I}}{4} \left( \frac{(\lambda_0 + 2\mu_0)M}{N} - 1 \right), \quad (15) \]
where
\[ M = \lambda_0 + \mu_0 + (3 + \lambda_0/\mu_0)(\alpha_2 + \beta_2), \]  
(16)
\[ N = (\mu_0 + \alpha_1 + \alpha_2)M - \beta_1^2(3 + \lambda_0/\mu_0). \]  
(17)

The quadropole elements for \( m = m' = 2 \) are
\[ T_{12,12}^{ss} = \left( \frac{k_s^2 a}{k_p} \right)^2 L_2, \]  
(18)
\[ T_{12,22}^{ss} = T_{22,12}^{ss} = (k_s a)^2 L_2, \]  
(19)
\[ T_{22,22}^{ss} = (k_p a)^2 L_2, \]  
(20)
where
\[ L_2 = \frac{i \pi}{4N} [\beta_1^2 + (\mu_0 + \alpha_1 + \alpha_2)(\mu_0 - \alpha_2 - \beta_2)]. \]  
(21)

The elements for \( m = 0 \) and \( m' = 2 \) are equal to those for \( m = 2 \) and \( m' = 0 \) and are coupling elements between monopole and quadropole waves and become
\[ T_{20,22}^{ss} = T_{22,20}^{ss} = (k_p a)^2 L_0, \]  
(22)
\[ T_{20,12}^{ss} = T_{12,20}^{ss} = (k_s a)^2 L_0, \]  
(23)
where
\[ L_0 = \frac{i \pi \beta_1 (\lambda_0 + 2\mu_0)}{2\sqrt{2N}}. \]  
(24)

These elements of course vanish when the circle becomes isotropic because \( \beta_1 = 0 \) then. It is noted that none of the doubly symmetric T matrix elements depends on the shear modulus \( c_4 \) of the circle.

For the part that is symmetric with respect to the \( x \) axis and antisymmetric with respect to the \( y \) axis or vice versa only monopole terms \( m = 1 \) contribute. These T matrix elements are actually the same as for an isotropic circle and are independent of the stiffness constants of the circle. They depend linearly on the difference in density between the circle and the surrounding and are therefore not of interest here and are thus not given.

The doubly antisymmetric part has even \( m \), exactly as the doubly symmetric part, but now the \( m = m' = 0 \) T matrix elements is not of leading order and is therefore not given. Also the elements for \( m = 0 \) and \( m' = 2 \), which are the same as for \( m = 2 \) and \( m' = 0 \), are very small, going as \( (k_s a)^4 \), and are not given. The quadropole elements for \( m = m' = 2 \) become
\[ T_{12,12}^{aa} = \left( \frac{k_s^2 a}{k_p} \right)^2 K_2, \]  
(25)
\[ T_{12,22}^{aa} = T_{22,12}^{aa} = (k_s a)^2 K_2, \]  
(26)
\[ T_{22,22}^{aa} = (k_p a)^2 K_2, \]  
(27)
where
\[ K_2 = \frac{i \pi (\mu_0 - \alpha_2 + \beta_2)}{4M}. \]  
(28)
where $M$ is defined above in Eq. (16). It is noted that $\alpha_2 - \beta_2 = c_4$ so the scattering depends linearly on the difference in shear moduli between the circle and the surrounding (with only a weak dependence on the other stiffness constants of the circle).

In the limit when the material inside the circle becomes isotropic all the $T$ matrix elements of course reduce to the corresponding isotropic ones.

4. The attenuation coefficient

To estimate the attenuation only the simplest approach is used. All multiple scattering is neglected and the scattering by each grain is supposed to take place in an isotropic medium with the effective stiffness constants of the material (but ignoring the attenuation in the material). The scattering by each grain is estimated by its total scattering cross section, a measure (a length in 2D) which gives the part of the incident plane wave that is scattered by the grain and thus lost to the coherent wave. This gives an estimation in 2D of the attenuation coefficient $\alpha$ as (see Zhang and Gross (7))

$$\alpha = \frac{1}{2} n \sigma = \frac{c \sigma}{2\pi a^2}.$$  \hspace{1cm} (29)

Here $n$ is the number density of grains, $\sigma$ is the total scattering cross section, and $c$ is the relative density of grains. This formula for the attenuation is supposed to be valid only for dilute concentrations, typically $c < 0.05$ or less. As the grains fill the whole volume $c = 1$ is used. The rationale for this is that each grain scatters extremely little in the low frequency limit. This approach has been used also by others, see references in Stanke and Kino (1).

It is straightforward to calculate the total scattering cross section, see Varadan et al. (6). The result is that for an incoming P wave it is

$$\sigma^P = \frac{4k_p^2}{k_s^2} \sum_{\sigma m} \left( |f_{1\sigma m}|^2 + \frac{k_s}{k_p} |f_{2\sigma m}|^2 \right),$$ \hspace{1cm} (30)

and for an incoming S wave

$$\sigma^S = \frac{4}{k_s} \sum_{\sigma m} \left( |f_{1\sigma m}|^2 + \frac{k_s}{k_p} |f_{2\sigma m}|^2 \right).$$ \hspace{1cm} (31)

These are expressed in the expansion coefficients of the scattered wave $f_{\sigma m}$. Expressing these coefficients in terms of the $T$ matrix and the expansion coefficients of the incoming wave, performing an average over all directions of incidence (which is the same as a random orientation of the grains), and specializing to the anisotropic circle at low frequencies without contrast in density to the surrounding, this simplifies and gives the attenuation coefficients (with the notation from the preceding section)

$$\alpha^P = \frac{2}{\pi k_p d^2} \left( \frac{k_s}{k_p} |T_{20,22}^{ss}|^2 + 2|T_{12,22}^{aa}|^2 + 2|T_{12,22}^{ss}|^2 + 2 \frac{k_s}{k_p} \left[ |T_{22,22}^{aa}|^2 + |T_{22,22}^{ss}|^2 \right] \right),$$ \hspace{1cm} (32)
\[ \alpha^s = \frac{4}{\pi k_p a^2} \left( |T_{12,12}^{aa}|^2 + |T_{12,12}^{ss}|^2 + \frac{k_s}{k_p} \left[ |T_{22,12}^{aa}|^2 + |T_{22,12}^{ss}|^2 \right] \right). \quad (33) \]

Here all the T matrix elements are given explicitly in simple form in the preceding section.

There do not seem to exist other results in 2D to compare these attenuation coefficients with and 3D results are not directly comparable (the behaviour on frequency is for instance different). Many results are for cubic materials, in the 2D situation this implies \( c_1 = c_2 \) and thus \( \beta_1 = 0 \). The expressions then simplify, but the dependence on the stiffness constants are still more complicated than reported results in 3D, see the review in Stanke and Kino (1). However, it should be remembered that many results in the literature assume weak anisotropy, whereas the present results are valid for strong anisotropy as well.

5. Conclusions

The main result here is the derived attenuation coefficients for 2D P-SV waves in a grainy material with grains that may have strong anisotropy. The main limitation is that the grains must be small compared to the ultrasonic wavelengths and that multiple scattering between grains is neglected.

References