Acoustic Emission During Crack Growth by a Hybrid Semi-Analytical/BEM Model

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Abstract
Acoustic emission (AE) has proved to be an efficient technique for health monitoring of large structures. AE sources may include phenomena related to cracks, dislocations, friction, etc. Crack initiation and structural damage will generate elastodynamic waves in the solid. This information can then be used for monitoring the integrity of structures.

In this paper, we propose a hybrid strategy coupling semi-analytical models for the crack growth with a numerical Boundary Element Method (BEM) model for elastodynamic wave propagation. Semi-analytical models describe mechanical fields in a domain holding a growing crack. These fields are then used as boundary conditions in the numerical elastodynamic model. The choice of semi-analytical models for the AE source and a numerical model for the propagation is dictated by the scale difference encountered in real applications between cracks and structures. The plane wave Fourier decomposition method can be used to take account semi-infinite isotropic solid.

Keywords: Modelling and Simulation, finite element method (FEM), wave propagation, Acoustic Emission (AE), plane waves decomposition

1. Introduction:

Acoustic Emission (AE) has proved to be an efficient technique for health monitoring of large structures. AE sources may include phenomena related to cracks, dislocations, phase transformations, friction, etc. In particular, sites enduring crack initiation and structural damage will generate elastodynamic waves in the solid. This information can then be used for monitoring the integrity of structures. For instance, data such as the localization and the number of cracks may be retrieved.

In this paper, we propose a hybrid strategy coupling semi-analytical models for the crack growth with a numerical Boundary Element Method (BEM) model for elastodynamic wave propagation. Semi-analytical models describe mechanical fields in a domain holding a growing crack. These fields are then used as boundary conditions in the numerical elastodynamic model. The choice of semi-analytical models for the AE source and a numerical model for the propagation is dictated by the scale difference encountered in real applications between cracks (mm) and structures (m).

The source model is based either on Westergaard’s exact complex solutions or asymptotic crack-tip fields expansions. A Fourier transform is used to translate fields influenced by crack growth from time-domain to harmonic-domain. The propagation model uses a numerical indirect BEM method based on a variational formulation that avoids the problems of hypersingular kernel due to the integral representation.

2. Equation and Methods

2.1. Boundary Elements Method for elastodymanics
When dealing with the modelization of an AE problem, two main issues arise. The first one is the need to handle phenomena with dissimilar time-spacial scales: the fracture process has characteristics significantly smaller than those of the waves propagation. The second issue is linked to the geometric complexity of the structures to be studied. Hence, a complete
analytical procedure may not likely be available for the problem at stake. In this, numerical methods can provide a convenient alternative to modelize and solve the problem.

Despite its versatility and generality, the FEM is here disadvantaged with its need to generate a domain discretization encompassing the geometric details at different scales [1]. The boundary element method (BEM) is ideally suited for the numerical analysis of this problem thanks to its light discretizations requirements [2-7]. It has been applied also for half-planes [8-10]

The BEM has been successfully applied to dynamic crack analysis in homogeneous, isotropic, linear elastic solids for many years (e.g. [11-14]). In principle, frequency-domain BEM can be applied to transient dynamic analysis of cracked solids by using an inverse Fourier-transform. A comparative study of the time-domain, frequency domain, and Laplace-domain BEM for transient dynamic crack analysis can be found in [15, 16].

Two-dimensional (2D) frequency-domain BEM using time-harmonic elastodynamic fundamental solutions have been presented by Dineva et al. [11] and Garcia-Sanchez et al. [12] for dynamic crack analysis.

The Navier-Cauchy equation for elastodynamics without body force is given by:

\[
\left( c^2_p - c_s^2 \right) u_{j,i,j} (x,t) + c_s^2 u_{i,j,i} (x,t) - \ddot{u}_j (x,t) = 0
\]

where commas and dots denote space and time differentiations respectively, \( u_j (x,t) \) represents the displacement vector at a field point \( x \) and at time \( t \), 

\[
c^2_p = \frac{\lambda + 2\mu}{\rho} \quad \text{and} \quad c_s^2 = \frac{\mu}{\rho}
\]

Lamé constants and \( \rho \) is the mass density.

The Fourier transform of the governing gives the following frequency domain representation:

\[
\left( c^2_p - c_s^2 \right) \tilde{u}_{j,i,j} (x,\omega) + c_s^2 \tilde{u}_{i,j,i} (x,\omega) + \omega^2 \tilde{u}_j (x,\omega) = 0
\]

The displacement field may be described by an integral representation as follows:

\[
\alpha(x) u_j (x) = \int_{\partial\Omega} G_{ij} (x,y,\omega) \sigma_{kj} (y) n_k (y) d\Gamma (y) - c.p.v. \int_{\partial\Omega} T_{ij} (x,y,\omega) u_j (y) d\Gamma (y)
\]

where:

\[
G_{ij} = \frac{i}{4\mu} \left[ \delta_{ij} H_0^l (k_s r) + \frac{1}{k_s^2} \left( H_0^l (k_s r) - H_0^l (k_l r) \right) \right]
\]

and

\[
T_{ij} = (\sigma_{j,k} (G_{jk} (x,y)) n_k (y)) e_i
\]

An integral representation for the stress can be derived as well:

\[
\alpha(x) \sigma_{ij} (x) = c.p.v. \int_{\partial\Omega} D_{ijk} (x,y,\omega) \sigma_{kl} (y) n_l (y) d\Gamma (y) - h.f.p. \int_{\partial\Omega} S_{ijk} (x,y,\omega) u_k (y) d\Gamma (y)
\]
where 

\[
\alpha(x) = 1, \text{ if } x \in \Omega \setminus \Gamma \\
\alpha(x) = \frac{1}{2}, \text{ if } x \in \Gamma \\
\alpha(x) = 0, \text{ if } x \not\in \Omega
\]

and \( G_{ij}, T_y, S_{jk}, D_{yk} \) are the displacement, the traction and the higher-order fundamental solutions for Navier-Cauchy equation.

When an integral is preceded with \( c.p.v. \) and \( h.f.p. \), it implies that it has to be evaluated in the sense of the Cauchy principal value and the Hadamard finite part respectively.

If we consider the specific case where tractions are given on the boundary (note that traction \( t_i = \sigma_{ij} n_j \) is assumed to be a known function continuous across the boundary \( \Gamma \)), the displacement \( u_i(x) \) and the stress \( \sigma_{ij}(x) \) may be represented by indirect integral representations (displacement discontinuity formulation):

\[
\begin{align*}
\sigma_{ij}(x) &= \int_{\Gamma} D_{jk}(x,y) d_k(y) \, d\Gamma(y) \\
u_i(x) &= c.p.v. \int_{\Gamma} T_y(x,y) d_j(y) \, d\Gamma(y)
\end{align*}
\]

The given boundary tractions are defined according to:

\[
t_i(x) = \sigma_{ij}(x) n_j(x) = h.f.p. \int_{\Gamma} S_{jk}(x,y) n_j(x) d_k(y) \, d\Gamma(y)
\]

where \( t_i(x) \) is the traction vector, \( n_i(x) \) denotes the outward unit normal vector on \( \Gamma \) at point \( x \) and \( d_k = \left\| u_k^+ - u_k^- \right\| \) represents the jump of the displacement vector.

Figure 1. Geometrical configuration

Usually, this equation is solved by using a collocation technique. However, such an approach has two disadvantages: the double layer term contains a singular integral which is difficult to compute, and the final algebraic system is not necessarily symmetric.

To overcome these two difficulties, we propose to associate a variational formulation to this system (the Galerkin-weighted residual method). The first advantage of this formulation is to avoid the explicit calculations of the “finite part” of singular integrals. A second advantage is the symmetrical nature of the linear implicit algebraic system obtained after discretization:
\[
\int_{\Gamma} \varphi_i(x) t_j(x) \, dx = \int_{\Gamma \times \Gamma} \varphi_i(x) D_{ijk}(x,y) n_j(x) d_k(y) \, d\Gamma(y) \, d\Gamma(x) = L(\tilde{\varphi}) = A(\tilde{\varphi}, \tilde{u})
\]  

(7)

where \( \tilde{\varphi}(x) \) represents a regular test function defined on \( \Gamma \), \( L(\tilde{\varphi}) \) is a linear form and \( A(\tilde{\varphi}, \tilde{u}) \) is a symmetrical bilinear form.

The double layer interaction term is hyper-singular and will be developed according to the transformation submitted in [13] and summarized in [3].

The boundary \( \Gamma \) is divided into \( N \) linear elements. A piecewise linear variation along each boundary element is assumed for the jumps of displacement and traction. The boundary displacement and traction along an element is interpolated with linear shape functions. A numerical scheme using three gauss points on each element is adopted to perform integration.

Once the transformation of singular integrals is achieved, the numerical integration is performed subdividing the current boundary element into sub-elements and dealing with each of them independently.

Finally, the algebraic system can be written in a symmetric matrix form as:

\[
[ID]\{d_i\} = \{\tau_i\}
\]  

(8)

The fact that the problem is formulated using the jump of displacements is not only useful for the evaluation of integrals. In fact, a classical BEM formulation would have required the definition of a specific subdomain surrounding the crack which would have led to spurious modes.

2.2. **Acoustic Emission source model**

The main assumption made for the simulation of fracture induced AE events is the uncoupled nature of the source and propagation models. The AE source model considers the mechanical perturbations induced by the crack growth and translates it into evolutive boundary conditions for the propagator. Here, the elastodynamic wave’s propagation is handled with a BEM module that appears suitable for the problem considered. However, if another propagator has to be used, the same AE source model may be kept if it still provides adequate boundary-conditions.

For the aimed applications, a single crack lying in an infinite plane domain subjected to remote uniform loads is considered. The crack-tips positions \( \pm a(t) \) are defined by a prescribed time function of general nature. Two fracture modes are taken into account: the opening one (mode I) and the in-plane sliding one (mode II). Mixed mode problems can be dealt with using a simple superposition of mode I/II solutions. The hypothesis of Linear Elastic Fracture Mechanics is assumed as well as the state of 2D plane strain.
The main hypothesis regarding the complete elastodynamic fracture problem is the quasi-static nature of the crack growth. With the time evolution of $a(t)$, the source model provides pseudo-dynamic values for the mechanical fields of interest. These quantities then serve as boundary conditions for the BEM elastodynamic solver. This hypothesis is reasonable as long as the crack growth speed does not exceed one third of Rayleigh wave speed. Solutions in the literature involving dynamic stress intensity factors may be more correct regarding the dynamics of the problem but lack of accuracy for points not located in the vicinity of crack-tips.

For the given fracture configuration, the exact complex solution of Westergaard [17] enables the accurate evaluation of the mode I stress:

$$\sigma_{11}(z) = 2 \text{Re} \left[ \phi_1^*(z) \right] - 2x \text{Im} \left[ \phi_1^*(z) \right] + C_1$$

$$\sigma_{22}(z) = 2 \text{Re} \left[ \phi_1^*(z) \right] - 2x \text{Im} \left[ \phi_1^*(z) \right] + C_1$$

$$\sigma_{12}(z) = -2x \text{Re} \left[ \phi_1^*(z) \right]$$

the displacements:

$$2\mu u_1'(z) = (\kappa - 1) \text{Re} \left[ \phi_1(z) \right] - 2x \text{Im} \left[ \phi_1(z) \right] + C_1 x$$

$$2\mu u_2'(z) = (\kappa + 1) \text{Im} \left[ \phi_1(z) \right] - 2x \text{Re} \left[ \phi_1(z) \right] - C_1 x$$

And the displacement jump on the crack:

$$[u_\frac{1}{2}](x,0) = \frac{\sigma_{22}^\infty (\kappa + 1)(a^2 - x_1^2)}{2\mu}$$

These expressions involve the following complex potential as well as its derivative and primitive:

$$\phi_1(z) = \frac{\sigma_{22}^\infty}{2} \left[ \frac{z}{(z^2 - a^2)^{1/2}} \right] + (\alpha - 1) \frac{\sigma_{22}^\infty}{4}$$
Expressions for the mode II solutions are given as follows for the stress:

\[
\begin{align*}
\sigma_{11}^2(z) &= 4 \text{Re}\left[\phi'_2(z)\right] - 2x_2 \text{Im}\left[\phi'_2(z)\right] \\
\sigma_{22}^2(z) &= 2x_2 \text{Im}\left[\phi'_2(z)\right] \\
\sigma_{12}^2(z) &= -2 \text{Im}\left[\phi_2(z)\right] - 2x_2 \text{Re}\left[\phi'_2(z)\right] - C_2
\end{align*}
\] (13)

The displacements:

\[
\begin{align*}
2\mu u_1^2(z) &= (\kappa + 1) \text{Re}\left[\phi_2(z)\right] - 2x_2 \text{Im}\left[\phi'_2(z)\right] - C_2 x_2 \\
2\mu u_2^2(z) &= (\kappa - 1) \text{Im}\left[\phi_2(z)\right] - 2x_2 \text{Re}\left[\phi'_2(z)\right] - C_2 x_1
\end{align*}
\] (14)

With the mode II complex potential defined as:

\[
\phi'_2(z) = -i \frac{\sigma_{12}^\infty}{2} \left[\frac{z}{(z^2 - a^2)^{\frac{1}{2}}}\right] + i \frac{\sigma_{12}^\infty}{2}
\] (15)

Where: \( C_2 = -\sigma_{12}^\infty \)

The availability of a complex solution is not mandatory. If a solution of satisfactory accuracy exists for the boundary condition fields, it may be used to represent the dynamic crack growth. The authors have shown that the stress field for the reference configuration can be described in the entire plane with series representations [17]. A classical Williams’s series can be used near crack tips and a Laurent series on the complementary space.

The availability of the analytical solution for crack displacement jump and a BEM integral formulation based on the same quantity is really convenient. The BEM mesh only needs to be defined on the crack support at its final extension. A classical BEM formulation would have required meshing a kind of box frame around the final crack. This would have led to the creation of spurious cavity modes.

With the source model defined in time domain while the BEM model is harmonic, Fourier transforms and inverse Fourier transforms are performed. A first FT provides harmonic boundary conditions from the source model. The harmonic BEM solution is then translated into time domain using an IFT so as to simulate waves propagation.

### 3. Results and discussion

All the subsequent results are performed for a model made of steel
(\( E = 210\text{GPa}, \eta = 0.3, \text{and} \rho = 7800\text{kg} / \text{m}^3 \)).

The crack initial length is \( 2a_0 = 2\text{mm} \) and its final length is \( 2a_f = 10\text{mm} \). The time step is \( \Delta t = 1.95E - 7 \text{ s} \).

#### 3.1. Influence of the Crack speed
The prescribed crack growth speed is here successively taken at 250, 500, 1000 m/s. The point where the displacement field is analyzed is given in polar coordinates by \( r = 0.75 \sqrt{2} \) m and \( \theta = \pi/4 \). The following results show the existence of two waves that propagate respectively at the dilatational wave speed (6020 m/s) and the shear wave speed (3218 m/s). The general shape of the two events remains the same as the crack growth speed increases. However, their dimensions change. The magnitude of the incoming perturbation increases with the speed. The duration of the event seems to decrease proportionally with the speed.

3.2. Directivity of the emission
An important issue in AE is the directivity of the emitting phenomenon. To assess this property for the crack growth source, the time evolution of displacements on an arc with \( \theta \in [0, \pi/2] \) at \( r = \sqrt{2} \times 0.75 \) m is observed. Displacements only differ from zero during two events propagating respectively at the dilatational wave speed (6020 m/s) and the shear wave speed (3218 m/s). In the following figure, the evolution of a displacement component is described on a time interval that includes the instant when the compressive wave is supposed to reach the points. The intensity appears to change significantly on the range of angles. For some points, the value remains constant while for some others it oscillates strongly. The intensity for the point \( \theta = 90^\circ \) remains equal to 0 which is consistent with the symmetries of the problem for the dilatational wave.
3.3. 2D waves propagation

The qualitative radial and angular behaviour of the source is at last assessed with the simulation of the elastodynamic response on a disk. It appears that two waves are emitted from the crack. They travel respectively at the dilatational wave speed (6020 m/s) and the shear wave speed (3218 m/s). The dilatational wave has the required symmetry properties while the shear wave shows the expected anti-symmetry. The intensity of the events decreases as they propagate.

Thanks to the BEM formulation, the response can be post-treated at any point (even at a far one). This feature is a clear advantage of the BEM over the FEM approach.
4. Conclusion

A procedure for the simulation of AE phenomena due to crack growth in 2D has been described. It combines a source model with a numerical elastodynamic solver. The effects of a given crack growth kinematic is translated into the time evolution of mechanical fields around or on the prescribed crack path. This information is then used as evolutive boundary conditions for the elastodynamic propagator. The source model being based on the exact quasi-static complex solution of Westergaard, it provides accurate description of the required fields. The use of a BEM solver in time-harmonic domain appears well adapted to the simulation of AE events in large structures. Thanks to this procedure, the effects of the crack growth parameters on the AE event can be assessed.

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