Analytical solutions for active and passive monitoring of structural damage in fibre-composite laminates

L R Francis Rose¹, Wing K Chiu², Nithurshan Nadarajah², Benjamin S Vien²

¹Aerospace Division, Defence Science and Technology Group,
Melbourne 3207, Australia francis.rose@dst.defence.gov.au
²Department of Mechanical & Aerospace Engineering, Monash University,
Clayton 3168, Australia

Abstract

The acoustic emission due to various forms of crack-like damage can be represented by an integral involving the emission due to appropriately defined crack elements, which in turn can be represented as particular combinations of force dipoles. These same fundamental solutions can also be used to represent the scattered field due to crack-like damage, which can be regarded as being due to an induced source at the damage location, activated by an incident wave. It is shown that the elastodynamic reciprocal theorem can be used in conjunction with a judiciously defined ansatz to derive a compact analytical form for these fundamental solutions in isotropic plates. The only calculations required are to determine the plane-wave dispersion curves and through-thickness mode functions. The radiation and scattering patterns for fibre breakage and delamination cracking are presented, based on the analytical solutions for Mode I and Mode II crack elements. These analytical results are compared with both finite element simulations and experimental measurements. The practical implications for active and passive structural health monitoring of composites are briefly discussed.

1. Introduction

Damage detection techniques can be broadly classified as active or passive, where the former rely on some form of interrogation of the damaged structure, and recording the response, whereas the latter rely on detecting signals generated by the damage process itself. Examples of active approaches include ultrasonics, X-rays, and optical techniques such as shearography [1-3]. The best-known passive approach is acoustic emission (AE) monitoring, which is primarily used as a qualitative technique for detecting and locating impact events [4]. However, there is currently a resurgence of research interest in quantitative AE monitoring, driven in part by recent advances in sensor technology, signal processing and computer modelling [5], but also in part by the attractiveness of in situ structural health monitoring (SHM) based on built-in sensor networks [6,7].

The failure mechanisms in composites have been extensively documented [8]. The ones of principal interest as AE source mechanisms are fibre breakage, delaminations, transverse matrix cracking and splitting. These failure mechanisms all involve crack-like sources of AE. The resulting AE can be represented theoretically as a convolution of (i) a source function, characterised by the displacement discontinuity across the
crack, and (ii) the stress wave radiated by point-like crack elements, which in turn can be expressed as various combinations of force doublets (or dipoles) [9]. This representation can also be used to describe the scattered field generated by crack-like damage due to an ultrasonic probing wave, because the scattered wave can be regarded as being due to an induced source [10,11]. Thus, the same theoretical framework can be employed to model both active and passive ultrasonic approaches for detecting impact-induced damage.

However, the practical exploitation of this framework has been hampered by the lack of easy-to-use analytical formulae for the radiated field in plate-like structures. Mal and co-workers [12] have presented a computational scheme for modelling AE sources in laminates, but this involves the numerical evaluation of multiple two-dimensional wavenumber integrals, which constitutes an onerous computational task. Achenbach and Xu [13] have derived a compact solution for the elastodynamic response due to an interior point force in a plate, based on the reciprocal theorem, but their approach appears to be restricted to isotropic plates and the solutions for crack elements were not derived. Velichko and Wilcox [14] derived a far-field approximation for the response of a cross-ply laminate to surface loads, starting from a 2D solution based on reciprocity, but interior sources were not considered.

This paper presents compact analytical formulae for the AE due to various crack elements at arbitrary through-thickness locations in a plate. The approach builds on our recent work for bulk waves [15] which involves a more direct implementation of reciprocity than that used in [13,14]. The only detailed calculations required are to generate the dispersion curves and associated through-thickness mode functions. Detailed results are presented for the radiation patterns due to Mode I and Mode II crack elements, corresponding to fibre breakage and delaminations, respectively, and the practical implications for AE source identification are discussed. The correct formulation of a point scatterer (or point source) limit for delaminations is discussed and clarified. For simplicity, the present calculations are restricted to an isotropic plate, but it will be shown in a forthcoming paper that the approach can be extended to a cross-ply laminate.

2. Representation theorem: body force equivalents

Consider the configuration shown in Fig. 1 which involves a localised crack-like form of damage indicated by the shaded region. In an active approach for damage detection, the structure is interrogated by an incident wave that generates a scattered field, as indicated schematically in Fig. 1. Assuming a time-harmonic incident field with frequency \( \omega \), and a linear scattering response, the scattered field \( u^S(x) \) can be expressed in terms of the displacement discontinuity \( \Delta u(x) \) across the crack surface \( \Sigma \) as follows [9]:

\[
\begin{align*}
  u^S_i(x) &= \int_\Sigma \Delta u_i(\xi) \frac{\partial G_m(\xi;\xi)}{\partial \xi^j} n_j d\Sigma(\xi), \\
  \Delta u_i(\xi) &= u^i_0(\xi) - u^i(\xi),
\end{align*}
\]  

(1)
Figure 1. A plate with localised damage indicated by shaded region. (a) Plate geometry and coordinate system, indicating an incident and a scattered wave; (b), (c) cross-sectional view of Mode I and Mode II cracks; (d), (e) body-force equivalents for the cracks.

where \( u^+ \) and \( u^- \) denote respectively the displacement on the upper and the lower side of the surface \( \Sigma \); \( n_j \) denotes the normal to that surface; \( c_{ijkl} \) denotes the local stiffness tensor; \( G_{ij}(\mathbf{x}; \xi) \) denotes the Green function representing the \( i^{th} \) component of displacement at \( \mathbf{x} \) due to a unit point force applied at \( \xi \) in the direction of the \( j^{th} \) axis, and the time dependence \( e^{-i\omega t} \) has been omitted.

Equation (1) can also serve to represent the radiated field due to crack-like sources of acoustic emission, i.e. according to Eq. (1), the scattered field can be regarded as being due to an induced source, characterised by the displacement discontinuity \( \Delta u \). Thus, the representation in Eq. (1) can be used for modelling both active and passive detection. In general, there are no analytical solutions for \( \Delta u \), but in the limit of small cracks compared with wavelength, the crack can be regarded as a point scatterer [11], or as a point source of AE [12]. In particular, we shall consider here two representative cases for composite laminates.

2.1 Mode I crack element: fibre breakage

First, fibre breakage can be modelled as a Mode I (tensile) crack element, with the crack surface lying in the \( yz \)-plane, as indicated in Fig. 1 (b). In the point-source limit, the source term can be specified in terms of a displacement discontinuity as follows:
\[ \Delta u_x = \Delta U \delta(y) \delta(z - \zeta) \] on \( x = 0, \)
\[ \Delta u_y = \Delta u_z = 0, \]

where the source strength \( \Delta U \) represents the crack volume, i.e. the integral of the crack opening \( \Delta u_x \) over the crack domain \( \Sigma \), and \( \delta \) denotes the Dirac delta function [16]. This source strength varies with time, reflecting the dynamics of crack growth, or of wave scattering. However, here, a time harmonic dependence is assumed, which can serve as a building block for arbitrary time dependence. On substituting Eq. (2) into Eq. (1), and assuming an isotropic material, i.e.

\[ c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \]

where \( \lambda, \mu \) denote the Lamé parameters and \( \delta_j \) the Kroenecker delta, one can derive the following body-force specification for this source term

\[ f_x = -\Delta U (\lambda + 2\mu) \delta'(x) \delta(y) \delta(z - \zeta), \]
\[ f_y = -\Delta U \lambda \delta(x) \delta'(y) \delta(z - \zeta), \]
\[ f_z = -\Delta U \lambda \delta(x) \delta(y) \delta'(z - \zeta), \]

where \( \delta' \) denotes the derivative of the delta function; these derivatives in the body force specification serve to generate force doublets (or dipoles), as indicated schematically in Fig. 1 (d). The field due to such doublets can be obtained by differentiating the Green function for a particular force direction with respect to the appropriate coordinate. However, it will be shown in the next section that the reciprocal theorem leads to a compact solution for the whole combination of doublets in Eq. (4).

2.2 Mode II crack element: delamination

Consider next a Mode II (shear) crack element located on plane parallel to the \( xy \)-plane, as indicated in Fig. 1 (c). This crack element can serve as a building block to model delamination damage, or as a point-scatterer (or point-source) approximation for sufficiently small delaminations. This source can be specified in terms of a displacement discontinuity, as follows:

\[ \Delta u_x = \Delta U \delta(x) \delta(y) \] on \( z = 0, \)
\[ \Delta u_y = \Delta u_z = 0. \]

The body force equivalent for this source is now given by

\[ f_x = -\Delta U \mu \delta(x) \delta'(y) \delta(z - \zeta), \]
\[ f_y = 0, \]
\[ f_z = -\Delta U \mu \delta'(x) \delta(y) \delta(z - \zeta). \]
3. Far-field solutions via reciprocal theorem

The reciprocal theorem for time-harmonic elastic fields can be stated as follows [17]:

\[
\int_V (\mathbf{f}^A \cdot \mathbf{u}^B - \mathbf{f}^B \cdot \mathbf{u}^A) dV = \int_S (\mathbf{T}^B \cdot \mathbf{u}^A - \mathbf{T}^A \cdot \mathbf{u}^B) dS,
\]

where \( \mathbf{u}, \mathbf{f} \) denote respectively the elastic displacement and body force for two states identified by the superscripts \( A, B \), and \( \mathbf{T} \) denotes the corresponding traction vector, i.e. \( \mathbf{T}_i = \sigma_{ij} \mathbf{n}_j \) where \( \sigma_{ij} \) denotes the stress tensor, and \( \mathbf{n} = \mathbf{n}_j \) the outward normal to the closed surface \( S \) surrounding the volume \( V \).

To apply this theorem in the present case, we first note that the far field radiation due to a localised source can be expressed as a sum over the propagating modes. In keeping with the viewpoint that guided waves can be regarded as thickness vibrations superimposed on a 2D carrier wave [18], an appropriate ansatz for the radiated field can be stated as follows:

\[
u^A_i(r, \theta, z) = \sum_n A^S_n(\theta)u^A_z(z; S_n)\Omega(k^S_n r) + \sum_n A^A_n(\theta)u^A_z(z; A_n)\Omega(k^A_n r),
\]

\[
u^B_i(r, \theta, z) = \sum_n A^S_n(\theta)u^B_z(z; S_n)\Omega(k^S_n r) + \sum_n A^A_n(\theta)u^B_z(z; A_n)\Omega(k^A_n r),
\]

\[
u^S_i(r, \theta, z) = \sum_n A^{SH}_n(\theta)u^S_z(z; SH_n)\Omega(k^{SH}_n r),
\]

\[
\Omega(kr) = \sqrt{\frac{i}{8\pi kr}} e^{ikr},
\]

where \( r, \theta, z \) denote cylindrical polar coordinates (cf. Fig. 1), \( k^S_n \) and \( u^A_z(z; S_n) \) denote respectively the wavenumber and through-thickness variation \( u^A_z \) for the \( n^{th} \) symmetric mode, with the \( x \)-axis as propagation direction, etc. The radial variation \( \Omega(kr) \) is chosen to be consistent with the appropriate 2D carrier wave, as discussed in [15]. Thus, it remains to determine the modal coefficients \( A^S_n, A^A_n, A^{SH}_n \) (for the symmetric, anti-symmetric and shear-horizontal modes, respectively) for any particular localised source.

For this purpose, the \( B \)-field is chosen to be a single mode propagating backwards at an angle \( \theta_0 \) with the \( x \)-axis, as indicated in Fig. 2, and the surface \( S \) is taken to consist of a circular cylindrical surface of radius \( R \) spanning the thickness \( 2h \) of the plate, completed by circular discs of radius \( R \) on the plate surfaces \( z = \pm h \).

### 3.1 Modal coefficients for Mode I crack element: fibre breakage

To describe the procedure in more detail, consider the case where the \( A \)-field is due to a Mode I crack element specified by Eq. (2), or by the body-force equivalent in Eq. (4),
and the $B$-field is a back-propagating symmetric mode $S_n$. It can then be shown, by following the procedure presented in [15] for the bulk wave case, that Eq. (7) leads to

$$\Delta U \sigma_{xx}^{B}(\xi; \theta_0; S_n) = \frac{2\pi n_S^S}{\omega k_n^S} A_n^S(\theta_0),$$

where $\sigma_{xx}^{B}(\xi)$ denotes the stress component of the $B$-field that is conjugate to the Mode I crack opening displacement from the viewpoint of the cross-work term in Eq. (7) evaluated at the location of the crack element, and $S_n^S$ denotes the cycle-averaged power flux for the $S_n$ mode. Thus, the modal coefficient $A_n^S$ in Eq. (8) is given by

$$A_n^S(\theta_0) = \Delta U \frac{\omega k_n^S}{2 \pi n_S^S} \sigma_{xx}^{B}(\xi; \theta_0; S_n).$$

The choice of notation makes it clear that corresponding formulae hold for the other modal coefficients, provided that the $B$-field is chosen as a back-propagating wave of the appropriate mode, which leads to

$$A_n^I(\theta_0) = \Delta U \frac{\omega k_n^A}{2 \pi n_A^A} \sigma_{xx}^{A}(\xi; \theta_0; A_n),$$

$$A_n^{SH}(\theta_0) = \Delta U \frac{\omega k_n^{SH}}{2 \pi n_{SH}} \sigma_{xx}^{SH}(\xi; \theta_0; SH_n).$$

Thus, it can be seen that the reciprocal theorem leads to explicit analytical formulae for the radiated field of a Mode I crack element at an arbitrary through-thickness location in a plate, with the modal coefficients in Eq. (8) being related to the conjugate stress component $\sigma_{xx}^{B}(\xi)$ of a suitably chosen $B$-field evaluated at the crack location.
3.2 Modal coefficients for Mode II crack element: delamination

The same procedure can be repeated for the Mode II crack element specified by Eq. (5), leading this time to the following modal coefficients,

\[ A_n^{S}(\theta_0) = \Delta U \frac{\omega_k^{S}}{2\eta_n^{S}} \sigma_{n}^{S}(\xi, \theta_0; S_n), \]

\[ A_n^{A}(\theta_0) = \Delta U \frac{\omega_k^{A}}{2\eta_n^{A}} \sigma_{n}^{A}(\xi, \theta_0; A_n), \]

\[ A_n^{SH}(\theta_0) = \Delta U \frac{\omega_k^{SH}}{2\eta_n^{SH}} \sigma_{n}^{SH}(\xi, \theta_0; SH_n). \]

These simple and intuitively plausible expressions for the modal coefficients of crack elements have not previously been reported. Their practical significance is discussed in the next section. It is noted that the approach presented above for an isotropic plate can be extended to a cross-ply laminate, albeit with some non-trivial subtleties that will be presented elsewhere.

4. Point source characteristics for laminate failure mechanisms

The current status of theoretical modelling and quantitative characterisation of AE source mechanisms in composites is extensively documented in [8-10]. This characterisation has generally focussed on features such as the risetime and amplitude of individual burst-like events, as a basis for AE source identification, often relying on a cluster analysis based on two or more characteristics. In particular, a recent cluster analysis based on (i) the extensional-to-flexural modal amplitude ratio, and (ii) the peak frequency, appears to be quite promising for identifying source mechanisms in simple laminates [19,20]. The underlying premise is that different failure mechanisms may exhibit distinctive AE modal signatures that reflect the failure dynamics (via the source risetime and amplitude), the fracture characteristics (e.g. Mode I versus Mode II), and the crack geometry, in particular the through-thickness location and crack size, thereby providing an adequate basis to identify and quantify the associated structural damage. The analytical solutions derived above constitute an essential pre-requisite for addressing this inverse problem of source identification. In particular, some valuable insights based on these solutions are presented here.

4.1 Point-source radiation pattern for Mode I crack element

Two important practical considerations can be deduced from the analytical solution from a Mode I crack element. First, the modal amplitudes are not isotropic: they exhibit a preferred direction, as shown in Fig. 3. This provides a possible basis for distinguishing between the AE due to transverse matrix cracking (in 90° plies) and axial splitting (in 0° plies), which both involve Mode I cracking, provided that the AE is recorded at a sufficient number of angular locations to allow the directivity pattern to be identified, contrary to current AE monitoring practice that often relies on a single sensor. Secondly, the directivity pattern that is measured in practice depends on the type
of sensor that is employed, and does not generally coincide with the modal radiation patterns shown in Fig. 3. Some examples for common sensor types will be presented elsewhere.

Figure 3. Radiation patterns. Azimuthal variation of modal amplitudes for the extensional and flexural modes due to a surface-breaking Mode I crack representing fibre breakage.

4.2 Point-scatterer representation for delaminations

Consider next the configuration shown in Fig. 4(a) showing an \( A_0 \) probing wave incident of a delamination. For simplicity, the delamination is assumed to be circular and to lie on the plate’s midplane. This configuration involves three characteristic lengths, viz. the wavelength \( \lambda \) of the incident wave; the delamination’s diameter \( d \); and the plate thickness \( 2h \). A point-scatterer approximation is appropriate provided that \( d << \lambda \). However, there are two distinct analytical formulations for this approximation depending on the ratio \( d/h \). For \( d << h \), the Mode II crack element discussed above is the appropriate point scatterer, as it has been assumed in previous work [12]. However, this relation is rarely applicable in practice. Instead, it is much more likely that \( h << d << \lambda \), in which case the correct point scatterer is more conveniently derived from a plate theory rather than from 3D elasticity.

Figure 4. Scattering amplitude due to a midplane delamination. (a) Configuration showing the delamination and incident wave; (b) azimuthal variation of scattering amplitude obtained computationally, compared with analytical point-scatterer approximations.
To clarify this point, Fig. 4(b) shows the $A_0$ scattering amplitude that is obtained computationally for the configuration shown in Fig. 4(a), with $2h = 3$ mm, $d = 10$ mm, and $\lambda = 30$ mm, corresponding to a frequency of 30kHz for the incident wave. Details of the computational model are the same as in our previous work [21]; this computational approach has been validated against experimental measurements for edge delaminations. It can be seen that the computational results agree closely with the results derived in [10] where a delamination is modelled as a flexural inhomogeneity (i.e. a region of lower flexural stiffness), within the framework of Mindlin plate theory. By contrast, the angular variation for a Mode II crack element does not show such good agreement with the computational results, thereby confirming the expectation above.

5. Conclusion

Remarkably simple analytical formulae have been derived for the radiated field due to Mode I or Mode II crack elements in an isotropic plate. The modal coefficients in these formulae have an intuitively clear interpretation as the conjugate stress associated with the auxiliary field that is employed when implementing the reciprocity relation. It is clear that the approach could also be used for arbitrarily oriented crack elements, with arbitrary displacement discontinuities across the crack, i.e. for all components of the moment tensor that is used for specifying crack-like AE sources [12]. These formulae have not previously been reported. They constitute a valuable pre-requisite for addressing the inverse problem of AE source identification. In particular, it has been noted that modal amplitudes generally vary with the observation direction, which indicates that it would be desirable in practice to obtain measurements from several observation directions for the purposes of source identification. The correct point-scatterer (or point source) representation for delaminations has also been discussed and clarified. An extension of the present work to fibre composite laminates will be reported elsewhere.

Acknowledgement

This work was funded by ARC Discovery Project DP 13101458.

References

17. JD Achenbach, Reciprocity in elastodynamics, CUP, Cambridge, UK 2003.