The state equations for the first and second fundamental problems of elastodynamics for a cracked medium.

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1. Introduction.

The equations of motion for a continuous medium with density ρ may be written as:

$$\begin{aligned}
&\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{\psi\psi}}{\partial \psi}+\frac{\partial \sigma_{xz}}{\partial z}=\rho\frac{\partial^2 u_x}{\partial t^2}\\
&\frac{\partial \sigma_{x\psi}}{\partial x}+\frac{\partial \sigma_{\psi\psi}}{\partial \psi}+\frac{\partial \sigma_{\psi z}}{\partial z}=\rho\frac{\partial^2 u_\psi}{\partial t^2}\\
&\frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{\psi z}}{\partial \psi}+\frac{\partial \sigma_{zz}}{\partial z}=\rho\frac{\partial^2 u_z}{\partial t^2}
\end{aligned} \tag{1}$$

In equation (1) we have not taken into account the body forces. Hooke's law may be written as

$$\begin{aligned}
&\sigma_{xx}=\lambda\theta+2\mu\varepsilon_{xx}, \quad \sigma_{\psi\psi}=\lambda\theta+2\mu\varepsilon_{\psi\psi}, \sigma_{zz}=\lambda\theta+2\mu\varepsilon_{zz} \qquad (2)\\
&\sigma_{x\psi}=2\mu\varepsilon_{x\psi}, \quad \sigma_{xz}=2\mu\varepsilon_{xz}, \quad \sigma_{\psi z}=2\mu\varepsilon_{\psi z},
\end{aligned}$$

The components of the strains are connected with the vector of displacement $\vec{u}$ according to:

$$\begin{aligned}
&\varepsilon_{xx}=\frac{\partial u}{\partial x}, \quad \varepsilon_{\psi\psi}=\frac{\partial v}{\partial \psi}, \quad \varepsilon_{zz}=\frac{\partial w}{\partial z} \qquad (3)\\
&\varepsilon_{x\psi}=\frac{1}{2}\left[\frac{\partial u}{\partial \psi}+\frac{\partial v}{\partial x}\right], \quad \varepsilon_{\psi z}=\frac{1}{2}\left[\frac{\partial v}{\partial z}+\frac{\partial w}{\partial \psi}\right], \quad \varepsilon_{xz}=\frac{1}{2}\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right]
\end{aligned}$$

We replace (2) and (3) into (1) and we get:

$$\mu\Delta u_x + (\lambda+\mu)\frac{\partial}{\partial x}(div\vec{u}) = \rho\frac{\partial^2 u_x}{\partial t^2}$$
$$\mu\Delta u_\psi + (\lambda+\mu)\frac{\partial}{\partial \psi}(div\vec{u}) = \rho\frac{\partial^2 u_\psi}{\partial t^2} \quad (4)$$
$$\mu\Delta u_z + (\lambda+\mu)\frac{\partial}{\partial z}(div\vec{u}) = \rho\frac{\partial^2 u_z}{\partial t^2}$$

The relations above may be written in vector form

$$\mu\Delta\vec{u} + (\lambda+\mu)\frac{\partial}{\partial x}grad(div\vec{u}) = \rho\frac{\partial^2 \vec{u}}{\partial t^2} \quad (5)$$

If we take into consideration the following condition:

$$grad(div\vec{u}) = rot(rot\vec{u}) + \Delta\vec{u} \quad (6)$$

And use it in Eq. (5) we find:

$$(\lambda+2\mu)grad(div\vec{u}) - \mu rot(rot\vec{u}) = \rho\frac{\partial^2 \vec{u}}{\partial t^2} \quad (7)$$

The displacement field of the elastic medium may be analyzed in a transverse and a longitudinal field. These two components of the displacement are propagating with different speeds that are independent. So we get:

$$\vec{u} = \vec{u}^p + \vec{u}^s \quad (8)$$

the components if which satisfy the equations:

$$rot\vec{u}^p = 0$$
$$div\vec{u}^s = 0 \quad (9)$$

Equation (7) based on (8) will result to

$$\frac{\partial^2 \vec{u}^p}{\partial t^2} = c_1^2 \left(\frac{\partial^2 \vec{u}^p}{\partial x^2} + \frac{\partial^2 \vec{u}^p}{\partial \psi^2} + \frac{\partial^2 \vec{u}^p}{\partial z^2} \right)$$
$$\frac{\partial^2 \vec{u}^s}{\partial t^2} = c_2^2 \left(\frac{\partial^2 \vec{u}^s}{\partial x^2} + \frac{\partial^2 \vec{u}^s}{\partial \psi^2} + \frac{\partial^2 \vec{u}^s}{\partial z^2} \right) \qquad (10)$$

where $c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}$, $c_2 = \sqrt{\frac{\mu}{\rho}}$ are the propagation speeds of the longitudinal (P) and the transverse (S) waves. In case when we consider the existence of body forces vector equation (5) will become:

$$\mu \nabla^2 \vec{u} + (\lambda + \mu) grad(div\vec{u}) + \rho \vec{P} = \rho \frac{\partial^2 \vec{u}}{\partial t^2} \qquad (11)$$

where $\vec{P}(X, \Psi, Z)$ is the vector of the body forces. If we write vector $\vec{\mathrm{P}}$ as

$$\vec{\mathrm{P}} = grad\Phi + rot\vec{\Psi} \qquad (12)$$

then we seek $\vec{u}$ as

$$\vec{u} = grad\phi + rot\vec{\psi} \qquad (13)$$

and the following condition should be satisfied

$$div\vec{u} = \nabla^2 \phi \qquad (14)$$

Vector equation (11) based on (12), (13) and (14) will become:

$$\mu \nabla^2 (grad\phi + rot\vec{\psi}) + (\lambda + \mu) grad(\nabla^2 \phi) + \rho grad\Phi + \rho rot\vec{\Psi} =$$
$$= \rho \frac{\partial^2}{\partial t^2}(grad\phi) + \rho \frac{\partial^2}{\partial t^2}(rot\vec{\psi}) \qquad (15)$$

$$grad[(\lambda+\mu)\nabla^2\phi+\rho\Phi-\rho\frac{\partial^2\phi}{\partial t^2}]+ \\ rot[\mu\nabla^2\vec{\psi}+\rho\vec{\Psi}-\rho\frac{\partial^2\vec{\psi}}{\partial t^2}]=0 \tag{16}$$

The form of Eq. (16) shows that the vector expression (13) will be the solution of the motion equation if and only if functions ϕ and $\vec{\psi}$ are selected in a way that they satisfy (be solutions of) the following equations

$$c_1^2\nabla^2\phi-\frac{\partial^2\phi}{\partial t^2}=-\Phi \tag{17}$$

$$c_2^2\nabla^2\vec{\psi}-\frac{\partial^2\vec{\psi}}{\partial t^2}=-\Psi \tag{18}$$

Equation (17) is the non-homogenous wave equation with wave propagation velocity c_1 which demonstrates that the component of the total displacement that corresponds to the gradient function ϕ is moving with c_1 speed. From (14) we get that the dilatation $\Delta = div\vec{u}$ satisfies the wave equation that corresponds to speed c_1. In seismology this wave is called primary or simply P-wave. This wave contributes to the changes in the volume of the medium.

On the other hand Eq. (18) shows that the other component of the boundary displacement, that corresponds to the vector function $\vec{\psi}$ is moving with a smaller speed, namely c_2. Using $rot\vec{u}=rot(rot\vec{\psi})$ we come up with the fact that $\vec{\omega}=\frac{1}{2}rot\vec{u}$ satisfies the wave equation with speed c_2. In seismology this wave is called secondary or simply S-wave.

This type of wave (transverse) refers to the twisting of the element without changes in its volume. In case when the shear modulus is zero then $c_2=0$. The above prove that transverse waves can't propagate in media with zero bending stiffness.

For our studies it would be useful to introduce the constant β, that is defined by:

$$\beta=\frac{c_1}{c_2}=\sqrt{\frac{\lambda+2\mu}{\mu}}=\sqrt{\frac{2-2\nu}{1-\nu}} \tag{19}$$

this constant is independent of the density and the elasticity modulus of the medium.

2. The method of Complex Analysis for two dimensional (2-D) Dynamic Problems.

Next, we will examine the solution of the two dimensional boundary problem in the frame of the dynamical theory of elasticity. We will limit the problem to the case of plane strain, where the following equations are valid:

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{x\psi}}{\partial \psi} &= \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial \sigma_{x\psi}}{\partial x} + \frac{\partial \sigma_{\psi\psi}}{\partial \psi} &= \rho \frac{\partial^2 v}{\partial t^2} \end{aligned} \qquad (20)$$

where:

$$\begin{aligned} \sigma_{xx} &= (\lambda + 2\mu)\frac{\partial u}{\partial x} + \lambda \frac{\partial v}{\partial \psi} \\ \sigma_{\psi\psi} &= \lambda \frac{\partial u}{\partial x} + (\lambda + 2\mu)\frac{\partial u}{\partial \psi} \\ \sigma_{x\psi} &= \mu\left(\frac{\partial u}{\partial \psi} + \frac{\partial v}{\partial x}\right) \end{aligned} \qquad (21)$$

and the condition of compatibility

$$\frac{\partial^2 \varepsilon_{xx}}{\partial \psi^2} + \frac{\partial^2 \varepsilon_{\psi\psi}}{\partial x^2} = \frac{\partial^2 \gamma_{x\psi}}{\partial x \partial \psi} \qquad (22)$$

where $\gamma_{x\psi} = 2\varepsilon_{x\psi}$

Hooke's Law (21) may be written as:

$$\begin{aligned} 2\mu \frac{\partial u}{\partial x} &= \sigma_{xx} - \frac{\lambda}{2(\lambda + \mu)}(\sigma_{xx} + \sigma_{\psi\psi}) \\ 2\mu \frac{\partial v}{\partial x} &= \sigma_{\psi\psi} - \frac{\lambda}{2(\lambda + \mu)}(\sigma_{xx} + \sigma_{\psi\psi}) \\ \theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial \psi} &= \frac{1}{2(\lambda + \mu)}(\sigma + \sigma_{\psi\psi}) \end{aligned} \qquad (23)$$

The compatibility condition (22) based on (23) will become:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial \psi}\right)(\sigma_{xx}+\sigma_{\psi\psi})-\frac{2(\lambda+\mu)}{\lambda+2\mu}\rho\frac{\partial^2}{\partial}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial \psi}\right) \tag{24}$$

or, based on (23_3) we get:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial \psi^2}-\frac{\rho}{\lambda+2\mu}\right)(\sigma_{xx}+\sigma_{\psi\psi})=0 \tag{25}$$

Equation (25) based on the definition of the speed c_1 of the P-wave will become:

$$\left(\nabla^2-\frac{1}{c_1}\frac{\partial^2}{\partial t^2}\right)(\sigma_{xx}+\sigma_{\psi\psi})=0 \tag{26}$$

If we differentiate equation (20_1) with respect to x and (20_2) with respect to ψ and then subtracting one from the other we get:

$$\left(\frac{\partial}{\partial x^2}-\frac{\rho}{2\mu}\frac{\partial^2}{\partial t^2}\right)\sigma_{xx}=\left(\frac{\partial^2}{\partial \psi^2}-\frac{\rho}{2\mu}\frac{\partial^2}{\partial \psi^2}\right)\sigma_{\psi\psi} \tag{27}$$

Based on the definition of the speed of the S-waves we will have:

$$\frac{\partial^2\sigma_{xx}}{\partial x^2}-\frac{\partial^2\sigma_{\psi\psi}}{\partial \psi^2}=\frac{1}{2c_2}\frac{\partial^2}{\partial \psi^2}(\sigma_{xx}+\sigma_{\psi\psi}) \tag{28}$$

Adding the results of the differentiations of equations (20) we get

$$\frac{\partial^2}{\partial x\partial \psi}(\sigma_{xx}+\sigma_{\psi\psi})+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial}-\frac{\rho}{\mu}\frac{\partial^2}{\partial t^2}\right)\sigma_{x\psi}=0 \tag{29}$$

ή $$\frac{\partial^2}{\partial x\partial \psi}(\sigma_{xx}+\sigma_{\psi\psi})+\left(\nabla^2-\frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right)\sigma_{x\psi}=0 \tag{30}$$

Relation (27) will be identically satisfies if we introduce the following relations for $\sigma_{xx}, \sigma_{\psi\psi}$

$$\begin{aligned} \sigma_{xx} &= \frac{\partial^2 U}{\partial \psi^2} - \frac{\rho}{2\mu}\frac{\partial^2 U}{\partial t^2} \\ \sigma_{\psi\psi} &= \frac{\partial^2 U}{\partial x^2} - \frac{\rho}{2\mu}\frac{\partial^2 U}{\partial t^2} \end{aligned} \tag{31}$$

and if we add them we will have:

$$\sigma_{xx} + \sigma_{\psi\psi} = \nabla^2 U - \frac{1}{c_2^2}\frac{\partial^2 U}{\partial t^2} \tag{32}$$

Based on (31) relations (25) and (23) will become:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \psi^2} - \frac{\rho}{\lambda + 2\mu}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \psi^2} - \frac{\rho}{\mu}\frac{\partial^2}{\partial t^2}\right)U = 0 \tag{33}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \psi^2} - \frac{\rho}{\mu}\frac{\partial^2}{\partial}\right)\left(\sigma_{x\psi} + \frac{\partial^2}{\partial x \partial \psi}\right) = 0 \tag{34}$$

$$\left(\nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\left(\nabla^2 - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right)U = 0 \tag{35}$$

$$\left(\nabla^2 - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right)\left(\sigma_{x\psi} + \frac{\partial^2 U}{\partial x \partial \psi}\right) = 0 \tag{36}$$

From the above its easy to understand that function U is the dynamic analogue of the Airy function. So the problem focuses on relation (35). If we concentrate on problems where the disturbance is propagating with speed c parallel to axis x, we can use the transformation:

$$\begin{cases} \xi = x - ct \\ n = \psi \end{cases} \tag{37}$$

Of course the general case $\xi = x \pm c_1^* t, \quad n = \psi \pm c_2^* t$, may be examined.

Relation (35) with respect to the new coordinates (ξ, n) will be:

$$\left(\frac{\partial^2}{\partial\xi^2}-\frac{1}{\mu_1^{*2}}\frac{\partial^2}{\partial n^2}\right)\left(\frac{\partial^2}{\partial\xi^2}-\frac{1}{\mu_2^{*2}}\frac{\partial^2}{\partial n^2}\right)U=0 \tag{38}$$

where μ_1^*, μ_2^* are the solutions of the characteristic equation.

$$\left(1-\frac{\rho c^2}{\lambda+2\mu}+\mu^{*2}\right)\left(1-\frac{\rho c^2}{\mu}+\mu^{*2}\right)=0 \tag{39}$$

$$\mu_1^*=i\left(1-\frac{\rho c^2}{\lambda+2\mu}\right)^{1/2}=i\left(1-\frac{c^2}{c_1^2}\right)^{1/2}=ia_1$$

$$\mu_2^*=i\left(1-\frac{\rho c^2}{\mu}\right)^{1/2}=i\left(1-\frac{c^2}{c_2^2}\right)^{1/2}=ia_2$$

Equations (31) and (32) through the new coordinates (ξ,n) will become:

$$\begin{aligned}\sigma_{xx}&=\frac{\partial^2 U}{\partial n^2}-\frac{c^2}{2c_1^2}\frac{\partial U}{\partial\xi^2}\\ \sigma_{\psi\psi}&=\frac{1}{2}(1+\alpha_2^2)\frac{\partial U}{\partial\xi^2}\end{aligned} \tag{40}$$

$$\sigma_{xx}+\sigma_{\psi\psi}=\frac{\partial^2 U}{\partial n^2}+a_2^2\frac{\partial^2 U}{\partial\xi^2} \tag{41}$$

The existence of two solutions forms two complex planes

$$\begin{aligned}z_1&=\xi+\mu_1^* n\\ z_2&=\xi+\mu_2^* n\end{aligned} \tag{42}$$

Using Eq. (42) in Eq. the solution will be found in the same way as for the case of the anisotropic medium:

$$U=F_1(z_1)+\overline{F_1(z_1)}+F_2(z_2)+\overline{F_2(z_2)}=2\,\mathrm{Re}[F_1(z_1)+F_2(z_2)] \tag{43}$$

where $F_i(z_i)\quad i=1,2,3$ are analytic functions with respect to complex variables z_j.

If we take into account relation (43), equations (43) and (41) will become:

$$\sigma_{xx} = -2\,\mathrm{Re}\left[\left(\frac{1}{2}+a_1^2-\frac{1}{2}a_2^2\right)F_1^{''}(z_1)+\frac{1}{2}(1+a_2^2)F_2^{''}(z_2)\right]= \quad (44)$$
$$= -2\,\mathrm{Re}\left[\left(a_1^2+\frac{1}{2}(1+a_2^2)\right)\Phi(z_1)+\frac{1}{2}(1+a_2^2)\Psi(z_2)\right]$$

$$\sigma_{\psi\psi} = (1+\alpha_2^2)\mathrm{Re}\left[F_1^{''}(z_1)+F_2^{''}(z_2)\right] = (1+a_1^2)\mathrm{Re}\left[\Phi(z_1)+\Psi(z_2)\right] \quad (45)$$

$$\sigma_{xx}+\sigma_{\psi\psi} = -2(\alpha_1^2-\alpha_2^2)\mathrm{Re}(F_1^{''}(z_1)) = -2(a_1^2-a_2^2)\mathrm{Re}\,\Phi(z_1)$$
(46)

Where $F_1^{''}(z_1)=\Phi(z_1)$ and $F_1^{''}(z_2)=\Psi(z_2)$. Using the above in Hooke's Law (21) or (23) and integrating we will get:

$$\mu u = -\mathrm{Re}\left[F_1^{'}(z_1)+\frac{1}{2}(1+a_2^2)F_2^{'}(z_2)\right] = -\mathrm{Re}\left[\phi(z_1)+\frac{1}{2}(1+a_2^2)\psi(z_2)\right] \quad (47)$$

$$\mu v = \mathrm{Im}\left[a_1F_1^{'}(z_1)+\frac{1+a_2^2}{a_2^2}F_2^{'}(z_2)\right] = \mathrm{Im}\left[a_1\phi(z_1)+\frac{1+a_2^2}{a_2^2}\psi(z_2)\right] \quad (48)$$

where $F_1^{'}(z_1)=\phi(z_1)$ and $F_2^{'}(z_2)=\psi(z_2)$.

Relation (21_3) based on the above equations will result to:

$$\sigma_{x\psi} = 2\,\mathrm{Im}\left[a_1F_1^{''}(z_1)+\frac{1+a_2^2}{4a_2^2}F_2^{''}(z_2)\right] = -2\,\mathrm{Re}\left[i\left(a_1\Phi(z_1)+\frac{1+a_2^2}{4a_2^2}\Psi(z_2)\right)\right] \quad (49)$$

Equations (44)-(49) gives us the components of the stress tensor and the displacement vector, in any point of the medium. These relations play the same role as the Kolosov – Muskhelisvilli – Lechnitski relations.

3.General method of solution of the 1st fundamental dynamic problem of elasticity for the cracked body.

For the first fundamental problem, we consider that the normal and tangential stresses on both of the crack lips are known. If we take into account the next relation:

$$2\left(\sigma_n + i\sigma_t\right) = \left(\sigma_{xx} + \sigma_{\psi\psi}\right) - e^{-2i\theta}\left[\sigma_{xx} - \sigma_{\psi\psi} + 2i\sigma_{x\psi}\right] \tag{50}$$

and the expressions for the components of stresses $\left(\sigma_{xx}, \sigma_{\psi\psi}, \sigma_{x\psi}\right)$ with respect to complex potentials $\Phi_1(z_1)$ and $\Psi_2(z_2)$ when $z \to t \in \ell,\ z_1 \to t_1, z_2 \to t$ they will become:

$$\sigma_{xx} + \sigma_{\psi\psi} = -\left(\alpha_1^2 - \alpha_2^2\right)\left[\Phi_1(t_1) + \overline{\Phi(t_1)}\right] \tag{51}$$

$$\sigma_{xx} - \sigma_{\psi\psi} = -\left(1 + \alpha_1^2\right)\left[\Phi_1(t_1) + \overline{\Phi_1(t_1)}\right] - \left(1 + \alpha_2^2\right)\left[\Psi_2(t_2) + \overline{\Psi_2(t_2)}\right] \tag{52}$$

$$\begin{aligned} \sigma_{x\psi} &= -2\,\mathrm{Re}\left\{ i\left[a_1\Phi(t_1) + \frac{\left(1 + a_2^2\right)}{4a_2}\Psi(t_2)\right]\right\} = \\ &= -\{\alpha_1\left[i\Phi_1(t_1) - i\overline{\Phi_1(t_1)}\right] + \frac{\left(1 + a_2^2\right)^2}{4\beta_2}\left[i\Psi(t_2) - i\overline{\Psi(t_2)}\right]\} \end{aligned} \tag{53}$$

and finally we will have:

$$\begin{aligned} 2\left[\sigma_n^{\pm} + i\sigma_t^{\pm}\right] &= -\left(a_1^2 - a_2^2\right)\left[\Phi_1^{\pm}(t_1) + \overline{\Phi^{\pm}(t_1)}\right] + \frac{\overline{dt}}{dt}\{\left(1 + a_1^2\right)\left(\Phi_1^{\pm}(t_1) + \overline{\Phi_1^{\pm}(t_1)}\right) + \\ &\left(1 + \alpha_2^2\right)\left(\Psi_2^{\pm}(t_2) + \overline{\Psi_2^{\pm}(t_2)}\right) + 2i\{a_1\left[i\Phi_1^{\pm}(t_1) - i\Phi_1^{\pm}(t_1)\right] + \frac{\left(1 + \alpha_2^2\right)}{4\alpha_2} + \\ &+ \left[i\Psi_2^{\pm}(t_2) - i\overline{\Psi_2^{\pm}(t_2)}\right]\}\}, \qquad \frac{dt}{\overline{dt}} = e^{-2i\theta} \end{aligned}$$

$$\begin{aligned} f^{\pm}(t) &= -\frac{\overline{dt}}{dt}\frac{1 + a_2^2}{2a_2}\left[(1 + a_2)^2\,\Psi_2^{\pm}(t_2) + (1 + a_2)^2\overline{\Psi_2^{\pm}(t_2)}\right] - \\ &- \left[-\left(a_1^2 - a_2^2\right) + \frac{\overline{dt}}{dt}(1 - a_1)^2\right]\Phi_1^{\pm}(t_1) + \left[-\left(\alpha_1^2 - \alpha_2^2\right) + \frac{\overline{dt}}{dt}(1 + a_1)^2\right]\overline{\Phi_1^{\pm}(t_1)} \end{aligned} \tag{54}$$

where $f^{\pm}(t) = 2\left(\sigma_n^{\pm} + i\sigma_t^{\pm}\right)$. If we multiply (54) with $dt/\overline{dt}$ and carry out the math we will get:

$$\begin{aligned}&2\left(1-\alpha_1\right)\frac{dt_1}{\overline{dt}}\Phi_1^{\pm}(t_1)+2\left(1+a_1\right)\frac{\overline{dt_1}}{dt}\overline{\Phi_1^{\pm}(t_1)}+\left(a_2-1\right)\frac{dt_1}{\overline{dt}}\left[\Phi_1^{\pm}(t_1)+\overline{\Phi_1^{\pm}(t_1)}\right]-\\&-\frac{\left(1+a_2^2\right)\left(1-a_2\right)^2}{2a_2}\Psi_2^{\pm}(t_2)+\frac{\left(1+a_2^2\right)\left(1+a_2\right)^2}{2a_2}\overline{\Psi_2^{\pm}(t_2)}=\frac{dt}{\overline{dt}}f(t)\end{aligned}\tag{55}$$

where: $\frac{dt_1}{dt}=\frac{1}{2}\left[(1+a_1)+(1-a_1)\frac{\overline{dt}}{dt}\right]$, $\frac{\overline{dt_1}}{\overline{dt}}=\frac{1}{2}\left[\left(1-a_1\right)+\left(1+a_1\right)\frac{dt}{\overline{dt}}\right]$

The conjugate expression of (55) becomes

$$\begin{aligned}&2\left(1-\alpha_1\right)\frac{\overline{dt_1}}{dt}\overline{\Phi_1^{\pm}(t_1)}+2\left(1+a_1\right)\frac{dt_1}{dt}\Phi_1^{\pm}(t_1)+\left(a_2-1\right)\frac{\overline{dt}}{dt}\left[\Phi^{\pm}(t_1)+\overline{\Phi^{\pm}(t_1)}\right]-\\&-\frac{\left(1+a_2^2\right)\left(1-a_2\right)^2}{2a_2}\overline{\Psi_2^{\pm}(t_2)}+\frac{\left(1+a_2^2\right)\left(1+a_2\right)^2}{2a_2}\Psi^{\pm}(t_2)=\frac{\overline{dt}}{dt}\overline{f^{\pm}(t)}\end{aligned}\tag{56}$$

We multiply relation (55) with $\frac{2\alpha_2}{\left(1+\alpha_2^2\right)\left(1+\alpha_2\right)^2}$ and relation (56) with $\frac{2\alpha_2}{\left(1+\alpha_2^2\right)\left(1-\alpha_2\right)^2}$ and consequently we add the results to find:

$$\begin{aligned}&\frac{8\alpha_2\left(1+\alpha_2^2\right)}{\left(1-\alpha_2^2\right)^2}\Psi^{\pm}(t_2)+\frac{2a_2}{\left(1+a_2^2\right)}\left[2\frac{1-a_1}{\left(1+a_2\right)^2}\frac{dt_1}{\overline{dt}}+2\frac{1+a_1}{\left(1-a_2\right)^2}\frac{dt_1}{dt}+\right.\\&\left.\frac{a_2-1}{\left(1+a_2\right)^2}\frac{dt}{\overline{dt}}+\frac{a_2-1}{\left(1-a_2\right)^2}\frac{\overline{dt}}{dt}\right]\Phi_1^{\pm}(t_1)+\frac{2a_2}{\left(1+a_2^2\right)}\left[2\frac{1+a_1}{\left(1+a_2\right)^2}\frac{\overline{dt_1}}{\overline{dt}}+\frac{2\left(1-a_1\right)}{\left(1-a_2\right)^2}\frac{\overline{dt_1}}{dt}+\right.\\&\left.+\frac{a_2-1}{\left(1+a_2\right)^2}\frac{dt}{\overline{dt}}+\frac{a_2-1}{\left(1-a_2\right)^2}\frac{\overline{dt}}{dt}\right]\overline{\Phi_1^{\pm}(t_1)}=g^{\pm}(t)\end{aligned}\tag{57}$$

where

$$g^{\pm}(t)=\frac{2a_2}{\left(1+a_2^2\right)\left(1+a_2\right)^2}\frac{dt}{\overline{dt}}f^{\pm}(t)+\frac{2a_2}{\left(1+a_2^2\right)\left(1-a_2\right)^2}\frac{\overline{dt}}{dt}f^{\pm}(t)$$

We multiply (57) with $\frac{\left(1-\alpha_2^2\right)^2}{8\alpha_2\left(1-\alpha_2^2\right)}$ and we have

$$\Psi^{\pm}(t_2)+A(t,\bar{t})\Phi_1^{\pm}(t_1)\frac{dt_1}{dt_2}+B_1(t,\bar{t})\overline{\Phi_1^{\pm}(t_1)}\frac{\overline{dt_1}}{dt_2}=g_1^{\pm}(t)\frac{dt}{dt_2} \tag{58}$$

where:

$$\begin{aligned}
A(t,\bar{t}) &= \frac{\left(1-a_2^2\right)^2}{4\left(1+a_2^2\right)^2}\left\{\frac{1}{\left(1+a_2\right)^2}\left[\left(1-a_1\right)^2+\frac{dt}{\overline{dt}}\left(a_2^2-a_1^2\right)\right]+\right. \\
&\left.+\frac{1}{\left(1-a_2\right)^2}\left[\left(1+a_1\right)^2+\frac{\overline{dt}}{dt}\left(a_2^2-a_1^2\right)\right]\right\} \left(\frac{dt_2}{dt}/\frac{dt_1}{dt}\right) \\
B(t,\bar{t}) &= \frac{\left(1-a_2^2\right)^2}{4\left(1+a_2^2\right)^2}\left\{\frac{1}{\left(1+a_2\right)^2}\left[\left(1+a_1\right)^2+\frac{dt}{\overline{dt}}\left(a_2^2-a_1^2\right)\right]+\right. \\
&\left.+\frac{1}{\left(1-a_2\right)^2}\left[\left(1-a_1\right)^2+\frac{\overline{dt}}{dt}\left(a_2^2-a_1^2\right)\right]\right\} \left(\frac{dt_2}{dt}/\frac{\overline{dt_1}}{dt_2}\right)
\end{aligned} \tag{59}$$

$$g_1^{\pm}(t)=\frac{\left(1-a_2^2\right)^2}{8a_2\left(1+a_2^2\right)}\frac{dt_2}{dt}$$

$$\frac{dt_2}{dt}=\frac{1}{2}\left[\left(1+a_2\right)+\left(1-a_2\right)\frac{\overline{dt}}{dt}\right]$$

$$\frac{dt_2}{\overline{dt}}=\frac{1}{2}\left[\left(1-a_2\right)+\left(1+a_2\right)\frac{dt}{\overline{dt}}\right]$$

As we know the complex potentials that describe the cracked body may be expressed via the Cauchy integrals, similarly with the case of the anisotropic medium:

$$\begin{cases}\Phi_1(z_1)=\dfrac{1}{2\pi i}\displaystyle\int_{\ell}\frac{\phi_2(\tau_1)}{\tau_1-z_1}dz_1 \\ \Psi_2(z_1)=\dfrac{1}{2\pi i}\displaystyle\int_{\ell}\frac{\psi(\tau_2)}{\tau_2-z_2}d\tau_2\end{cases} \tag{60}$$

Sohotsky – Plemely formulas when $z\to t\ (z_1\to t_1, z_2\to t_2)$ become:

$$\Phi_1^{\pm}(t_1) = \pm\frac{1}{2}\phi(t_1) + \frac{1}{2\pi i}\int_{\ell}\frac{\phi(\tau_1)}{\tau_1 - t_1}d\tau_1 \tag{61}$$

$$\Psi_2^{\pm} = \pm\frac{1}{2}\psi(t_2) + \frac{1}{2\pi i}\int_{\ell}\frac{\psi(\tau_2)}{\tau_2 - t_2}d\tau_2 \tag{62}$$

We subtract relations (58) and based on (61) and (62) we find:

$$\psi(t_2) + A\left(t,\bar{t}\right)\phi(t_1)\frac{dt_1}{dt_2} + B\left(t,\bar{t}\right)\overline{\phi(t_1)}\frac{\overline{dt_1}}{dt_2} = \lambda_1(t)\frac{dt}{dt_2} \tag{63}$$

where $\lambda_1(t) = g_1^+(t) - g_1^-(t)$

The complex potential $\Psi_2(z_2)$ based on (63) will have the form:

$$\begin{aligned}\Psi_2(z_2) = \frac{1}{2\pi i}\int_{\ell}\frac{\lambda_1(\tau)}{\tau_2 - z_2}d\tau - \frac{1}{2\pi i}\int_{\ell}\frac{A(\tau,\bar{\tau})\phi(\tau_1)}{\tau_2 - z_2}d\tau_1 - \\ - \frac{1}{2\pi i}\int_{\ell}\frac{B(\tau,\bar{\tau})\overline{\phi(\tau_1)}}{\tau_2 - z_2}\overline{d\tau_1}\end{aligned} \tag{64}$$

Adding relations (58) based on (61), (62) and the expressions (64) we will have the next singular integral equation:

$$\begin{aligned}A(t,\bar{t})\left(\frac{dt_1}{dt}/\frac{dt_2}{dt}\right)\frac{1}{\pi i}\int_{\ell}\frac{\phi(\tau_1)}{\tau_1 - t_1}d\tau_1 - B\left(t,\bar{t}\right)\left(\frac{\overline{dt_1}}{dt}/\frac{dt_2}{dt}\right)\frac{1}{\pi i}\int_{\ell}\frac{\overline{\phi(\tau_1)}}{\overline{\tau_1 - t_1}}\overline{d\tau_1} - \\ - \frac{1}{\pi i}\int_{\ell}\frac{A(\tau,\bar{\tau})\phi(\tau_1)}{\tau_2 - t_2}d\tau_1 - \frac{1}{\pi i}\int_{\ell}\frac{B(\tau,\bar{\tau})\overline{\phi(\tau_1)}}{\tau_2 - t_2}\overline{d\tau_1} = \left[g_1^+(t) + g_2^-(t)\right]\frac{dt}{dt_2} - \\ - \frac{1}{\pi i}\int_{\ell}\frac{\lambda_1(\tau)}{\tau_2 - t_2}d\tau\end{aligned} \tag{I}$$

Singular integral equation (I) is the state equation of the first fundamental dynamic problem for the cracked body in case when the stresses $\left(\sigma_n^{\pm}, \sigma_t^{\pm}\right)$ are known for both the crack lips.

4. The 2nd fundamental dynamic problem of elasticity for a cracked body.

For the second fundamental problem, we consider known the displacements on both the crack lips. We have:

$$2\mu\big(u(z)+iv(z)\big)=-\left[\beta_1\phi_1(z_1)+\beta_2\overline{\phi_1(z_1)}+\beta_3\psi_2(z_2)+\beta_4\overline{\psi_2(z_2)}\right] \quad (65)$$

where: $\beta_1=1+\alpha_1, \beta_2=1-\alpha_2 \quad \beta_3=\frac{1}{2}\left(1+\alpha_2^2\right)\left(1+\frac{1}{\alpha_2}\right)$ και $\beta_4=\frac{1}{2}\left(1+\alpha_2^2\right)\left(1-\frac{1}{\alpha_2}\right)$

If we take the limiting values of expression (65) when $z\to t\in\ell \;\; (z_1\to t_1,\, z_2\to t_2)$ and differentiatie with respect to variable t we get:

$$2\mu\frac{d}{dt}\left[u^{\pm}_{(t)}+iv^{\pm}_{(t)}\right]=-\left[\beta_1\frac{dt_1}{dt}\Phi_1^{\pm}(t_1)+\beta_2\frac{\overline{dt_1}}{dt}\overline{\Phi_1^{\pm}(t_1)}+\beta_3\frac{dt_2}{dt}\Psi_2^{\pm}(t_2)+\beta_4\frac{\overline{dt_2}}{dt}\overline{\Psi_2^{\pm}(t_2)}\right] \quad (66)$$

$$\frac{2\mu}{\beta_4}\frac{dt}{\overline{dt_2}}\frac{d}{dt}\left(u^{\pm}_{(t)}+iv^{\pm}_{(t)}\right)=-\left[\frac{\beta_1}{\beta_4}\frac{dt_1}{\overline{dt_2}}\Phi_1^{\pm}(t_1)+\frac{\beta_2}{\beta_4}\frac{\overline{dt_1}}{\overline{dt_2}}\overline{\Phi_1^{\pm}(t_1)}+\frac{\beta_1}{\beta_4}\frac{dt_2}{\overline{dt_2}}\Psi^{\pm}(t_2)+\overline{\Psi^{\pm}(t_2)}\right] \quad (67)$$

An analogous expression of (65) is the following:

$$2\mu\big(u(z)+iv(z)\big)=-\left[\beta_2\phi(z_1)+\beta_1\overline{\phi_1(z_1)}+\beta_4\psi_2(z_2)+\beta_3\overline{\psi(z_2)}\right] \quad (68)$$

For $z\to t$ and consequently differentiation with respect to t , we shall get after carrying out the mathematics:

$$\frac{2\mu}{\beta_3}\frac{dt}{\overline{dt_2}}\frac{d}{dt}\left(u^{\pm}_{(t)}+iv^{\pm}_{(t)}\right)=-\left[\frac{\beta_2}{\beta_3}\frac{dt_1}{\overline{dt_2}}\Phi_1^{\pm}(t_1)+\frac{\beta_1}{\beta_3}\frac{\overline{dt_1}}{\overline{dt_2}}\overline{\Phi_1^{\pm}(t_1)}+\frac{\beta_4}{\beta_3}\frac{dt}{\overline{dt_2}}\Psi^{\pm}(t_2)+\Psi^{\pm}(t_2)\right] \quad (69)$$

We subtract relation (69) from relation (67) and we get:

$$(\beta_3-\beta_4)2\mu\frac{dt}{dt_2}\frac{d}{dt}\left(u^{\pm}_{(t)}-iv_{(t)}\right)=-\left[(\beta_1\beta_3-\beta_4\beta_2)\frac{dt_1}{dt_2}\Phi_1^{\pm}(t_1)+\right.$$
$$\left.+(\beta_2\beta_3-\beta_1\beta_4)\overline{\frac{dt_1}{dt_2}}\overline{\Phi_1^{\pm}(t_1)}+(\beta_3^2-\beta_4^2)\frac{dt_2}{dt_2}\Psi^{\pm}(t_2)\right] \tag{70}$$

We multiply equation (70) with $\dfrac{1}{\beta_3^2-\beta_4^2}\dfrac{\overline{dt_2}}{dt_2}$ and we result to:

$$2\mu\frac{dt}{dt_2}\frac{\beta_3-\beta_4}{\beta_3^2-\beta_4^2}\frac{d}{dt}\left(u^{\pm}_{(t)}+iv^{\pm}_{(t)}\right)=-\left[\frac{\beta_1\beta_3-\beta_4\beta_2}{\beta_3^2-\beta_4^2}\frac{dt_1}{dt_2}\Phi_1^{\pm}(t_1)+\right.$$
$$\left.+\frac{\beta_2\beta_3-\beta_1\beta_4}{\beta_3^2-\beta_4^2}\frac{\overline{dt_1}}{dt_2}\overline{\Phi_1^{\pm}(t_1)}+\Psi^{\pm}(t_2)\right] \tag{71}$$

Αφαιρώντας τις σχέσεις (71) μεταξύ τους και λαμβάνοντας υπ' όψη τους τύπους Sohotsky – Plemely θα έχουμε:

$$\psi(t_2)=-2\mu\frac{d}{dt}\left[(u^+-u^-)-i(v^+-v^-)\right]\frac{(\beta_3-\beta_4)}{\beta_3^2-\beta_4^2}\frac{dt}{dt_2}-$$
$$-\frac{\beta_1\beta_3-\beta_4\beta_2}{\beta_3^2-\beta_4^2}\frac{dt_1}{dt_2}\phi_1(t_1)-\frac{\beta_3\beta_2-\beta_1\beta_4}{\beta_3^2-\beta_4^2}\frac{\overline{dt_1}}{dt_2}\overline{\phi_1(t_1)} \tag{72}$$

Θέτουμε:

$$\lambda_1^*(t)=2\mu\frac{d}{dt}\left[\left(u^+-u^-\right)+i\left(v^+-v^-\right)\right]\frac{\beta_3-\beta_4}{\beta_3^2-\beta_4^2}$$

Συνεπώς, η έκφραση του μιγαδικού δυναμικού $\Psi_2(z_2)$ βάση της έκφρασης (72) θα πάρει την μορφή:

$$\Psi_2(z_2)=\frac{1}{2\pi i}\int_{\ell}\frac{\psi(\tau_2)}{\tau_2-z_2}d\tau_2-\frac{1}{2\pi i}\int_{\ell}\frac{\lambda_1^*(\tau)}{\tau_2-z_2}d\tau_2-\frac{\beta_1\beta_3-\beta_4\beta_2}{2\pi i}\int_{\ell}\frac{\phi_1(z_1)}{\tau-z_2}d\tau_1-$$
$$-\frac{\beta_3\beta_2-\beta_1\beta_4}{2\pi i}\int_{\ell}\frac{\overline{\phi_1(\tau_1)}}{\tau_2-z_2}\overline{d\tau_1} \tag{73}$$

Προσθέτοντας τις σχέσεις (71) και λαμβάνοντας υπ' όψη τις οριακές τιμές τύπου Sohotsky – Plemely των μιγαδικών δυναμικών $\Phi_1(z_1)$ και $\Psi_2(z_2)$ θα έχουμε την παρακάτω ιδιόμορφη ολοκληρωτική εξίσωση:

$$
\begin{aligned}
&\frac{(\beta_4\beta_2-\beta_1\beta_3)}{\pi i}\frac{dt_1}{dt_2}\int_\ell \frac{\phi_1(\tau_1)}{\tau_1-t_1}d\tau_1+\frac{\beta_2\beta_3-\beta_1\beta_4}{\pi i}\frac{\overline{dt_1}}{dt_2}\int_\ell \frac{\phi_1(\tau_1)}{\overline{\tau_1}-\overline{t_1}}\overline{d\tau_1}+\\
&+\frac{\beta_4\beta_2-\beta_1\beta_3}{\pi i}\int_\ell \frac{\phi(\tau_1)}{\tau_2-t_2}d\tau_1+\frac{\beta_1\beta_4-\beta_3\beta_2}{\pi i}\int_\ell \frac{\overline{\phi_1(\tau_1)}}{\tau_2-t_2}\overline{d\tau_1}= \qquad \text{(II)}\\
&=\lambda_2^*(t)-\frac{1}{\pi i}\int_\ell \frac{\lambda_1^*(\tau)}{\tau_2-t_2}d\tau
\end{aligned}
$$

όπου:

$$\lambda_2^*(t)=2\mu\frac{d}{dt}\left[\left(u^+ + u^-\right)+i\left(v^+ + v^-\right)\right](\beta_3-\beta_4)\frac{dt}{dt_2}$$

Η εξίσωση (II) είναι η καταστατική εξίσωση του $2^{ου}$ θεμελιώδους δυναμικού προβλήματος της ελαστικότητας.