NUMERICAL SIMULATIONS OF ULTRASOUND NDE:
A HYBRID MODEL WITH IMPROVED EFFICIENCY BY A NEW BOUNDARY FORCING METHOD

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ABSTRACT

For ultrasonic NDE, modelling tools able to represent the entire domain are desirable. However, different tools are attractive for different tasks. The finite element (FE) method, for example, is good for scattering by complex shapes, but for large volumes the calculation becomes unwieldy. The ability to utilise suitable small models for different features (sources, scatterers, receivers etc), and link them together to perform accurate modelling with manageable sized models would be advantageous. The generic hybrid method takes output from one region and processes it to use as input to another separated region. Previous work at Imperial College on this prototype method tested the concept using the values of displacements and stresses on the boundary of region 1 to calculate the values of displacements/velocities and stresses elsewhere: in particular on the boundary of region 2. (This takes place in the frequency domain, but is applied in the time domain by using Fourier transforms and inverse transforms). This allows region 2 to be excited in the FE model by specifying time-series for either stress or displacement/velocity along a line in front of region 2. Here, we introduce an extension to the method by modifying the forcing presented to the second region. This allows suitable point forces to be applied to the boundary of the second region to reproduce the incident field within that region, without introducing numerical scattering at its boundary and hence allowing both the size of the modelled region to be reduced substantially and the extraction of the scattered field on the boundary to be simplified. Some animations demonstrating the successful test of this concept will be included. In a separate presentation some examples of industrial interest, calculated using this method, will be presented and discussed.

INTRODUCTION

The concept behind the hybrid model is that different numerical methods are most appropriate in different situations. The FE model is good for very detailed modelling near a complex feature. Analytical methods are good for open spaces. For examples containing some complex features (e.g., sources and defect) which are widely spaced the hybrid approach seeks to use appropriate small models near to the complex features and link them together using a hybrid interface, as illustrated schematically in figure 1, and discussed in [1,2], for example.

Figure 1 Simplified schematic geometry for source & defect: Whole region or Hybrid Interface
A GENERIC HYBRID MODEL

For simplicity of exposition only the 2-dimensional case is discussed in detail here. As it is convenient to work in the frequency domain, any output from the local modelling in the time domain is first subject to a FFT.

The generic hybrid model in the frequency domain for in-plane elastic motion is based here on a Green’s function approach, using the displacement potentials. In dimensionless variables the equations of motion may be written as

\[-\omega^2 u_x = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \]
\[-\omega^2 u_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} \]
\[\tau_{xx} = \gamma^{-2} \frac{\partial u_x}{\partial x} + (\gamma^{-2} - 2) \frac{\partial u_y}{\partial y} \]
\[\tau_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \]
\[\tau_{yy} = (\gamma^{-2} - 2) \frac{\partial u_x}{\partial x} + \gamma^{-2} \frac{\partial u_y}{\partial y} \]

in which \(\tau_{xx}, \tau_{xy}, \) and \(\tau_{yy}\) are the stress components, \(u_x\) and \(u_y\) are displacement components, \(\gamma = c_T/c_L\) is the ratio of the shear \((c_T)\) and compressional \((c_L)\) wave speeds in the elastic material, and \(\omega\) is the angular frequency of the time-harmonic motion. The displacement potentials are introduced via

\[u_x = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \]
\[u_y = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \]

and they satisfy the Helmholtz equations

\[(\nabla^2 + \omega^2)\psi = 0 \]
\[0 \]
\[(\nabla^2 + \omega^2\gamma^2)\phi = 0. \]

Green’s theorem allows us to use the values of the potential and its normal derivative on a closed curve to calculate the potential elsewhere. For example, for the shear potential

\[\psi(x_0) = -\oint_S (\nabla \phi) \cdot \mathbf{n} dS \]

in which \(\mathbf{n}\) is the outward normal to the contour \(S\) and \(G_{\psi}(x, x_0)\) is the Green’s function of the Helmholtz equation for \(\psi, \)

\[G_{\psi}(x, x_0) = -\frac{i}{4} H_0(\omega|x-x_0|) \]

and for the longitudinal potential

\[\Phi(x_0) = -\oint_S (\nabla \phi) \cdot \mathbf{n} dS \]
\[G_{\phi}(x, x_0) = -\frac{i}{4} H_0(\omega\gamma|x-x_0|). \]

Here, we make use of these properties to construct a hybrid model for scattering in a large region by defining two smaller areas, one containing the source (region 1) and the other containing the scatterer (region 2), and surrounding them with PML or absorbing material, [3], as shown in figure 1. Any numerical model appropriate to the source region can then be used to obtain suitable outputs on the boundary from which the potentials and their normal derivatives there, in the frequency domain, can be extracted and used, as above, to calculate values on the second boundary. For convenience we choose the boundary to be rectangular.
Having found the values for displacement and stress components on the second boundary in the frequency domain, inverse Fourier transforms may be applied to obtain the displacements, stresses and velocities there in the time domain. The problem then arises as to how to make use of this information to force a small numerical model of this second scattering region. One option is to specify either the velocity or the stress have the given values there. Both of these options however lead to problems: (i) Reflections will occur at the artificial boundary, so the calculation area must be large enough that these don’t have time to interfere with the extraction of the quantities required. (ii) This generates waves which travel away from the artificial boundary, which is acceptable on those faces which are between the source and the scatterer, but clearly unacceptable on those faces where the source and the scatterer are on the same side and in such cases an ad-hoc decision is made not to force on those sides.

A NEW BOUNDARY FORCING METHOD
The approach which we adopt here is different. We no longer require the values of the stress or velocity to have given values on the boundary. Instead we construct the times series for the forces around the artificial boundary which, when added to the system reproduce both the velocities and stresses of the incident wave field on the artificial boundary, and hence also the incident field within the calculation area. This procedure is summarised in the flow chart shown in figure 2. In order to achieve this, the time domain form of the elastic equations using velocity and stress are considered:

$\frac{\partial v_x}{\partial t} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}$

$\frac{\partial v_y}{\partial t} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}$

$\frac{\partial \tau_{xx}}{\partial t} = \gamma^{-2} \frac{\partial v_x}{\partial x} + (\gamma^{-2} - 2) \frac{\partial v_y}{\partial y}$

$\frac{\partial \tau_{xy}}{\partial t} = \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}$

$\frac{\partial \tau_{yy}}{\partial t} = (\gamma^{-2} - 2) \frac{\partial v_x}{\partial x} + \gamma^{-2} \frac{\partial v_y}{\partial y}$

We use a small finite difference scheme for the computation area, with no scatterer present and surrounded by a small PML, as shown in figure 3. As no scatterer is present there is no need for a detailed mesh for this part of the calculation.
The velocities on the boundary are known at times $n\delta t$ and the stresses at intermediate times $(n+1/2)\delta t$. The equations are integrated for one time step throughout the computation area. Thus, when time-stepping for the velocities

$$v_x((n+1)\delta t) = v_x(n\delta t) + \delta t \left( \frac{\partial \tau_{xx}}{\partial x} ((n + \frac{1}{2})\delta t) + \frac{\partial \tau_{xy}}{\partial y} ((n + \frac{1}{2})\delta t) \right)$$

$$v_y((n+1)\delta t) = v_y(n\delta t) + \delta t \left( \frac{\partial \tau_{xy}}{\partial x} ((n + \frac{1}{2})\delta t) + \frac{\partial \tau_{yy}}{\partial y} ((n + \frac{1}{2})\delta t) \right)$$

in which the spatial derivatives are obtained using centred finite differences. We require that both velocity components on the artificial boundary 2 should have the values $v^H(x, (n+1)\delta t)$. Thus, an additional velocity $v^A(x, (n+1)\delta t)$ must be added there:

$$v^A(x, (n+1)\delta t) = v^H(x, (n+1)\delta t) - v(x, (n+1)\delta t)$$

for $x$ on the boundary, and for both velocity components.

Next, we note that if we had instead added forces $F^A$ to the right hand side of those equations of motion it would have resulted in an additional velocity of $F^A\delta t$. Hence, our added velocities may be equated to an additional applied force:

$$F^A(x) = \frac{v^A(x)}{\delta t}.$$

These values for the forces which need to be applied to the boundary are recorded for later use and the values of the velocity components on the boundary are updated to $v^H(x, (n+1)\delta t)$ there, ready for the next step. In order to mimic the incident field inside the region both the velocities and the stresses on its boundary must have the values predicted by the hybrid theory. The analysis above demonstrates how to fix the velocity there. It now remains to fix the stress there too. This requires the normal gradients of the velocities to have the correct values there, and this can be achieved by adding suitable values to the velocities at points which are one grid point outside the boundary, and recording the equivalent additional forces to be added there.

For example, on faces with $x=$constant, updating $\tau_{xx}(x, (n + \frac{1}{2})\delta t)$ and $\tau_{yy}(x, (n + \frac{1}{2})\delta t)$ to achieve $\tau^H_{xx}(x, (n + \frac{1}{2})\delta t)$ and $\tau^H_{yy}(x, (n + \frac{1}{2})\delta t)$ requires an additional velocity $v^A_x(x \pm \delta x, (n+1)\delta t)$ to have been applied previously, where the $\pm$ is chosen to ensure this location is outside the boundary.

Figure 3 Finite Difference Grid for Applied Forces Calculation

![Finite Difference Grid for Applied Forces Calculation](image)
Similarly, updating $\tau_{xy}(x, (n + \frac{1}{2})\delta t)$ on this face requires an additional $v^A_x(x + \delta x, (n + 1)\delta t)$ to have been applied. These are found by time-stepping the equations

$$
\frac{\partial v_x}{\partial x} = \frac{1}{4(y^2 - 1)} \left( y^{-2} \frac{\partial \tau_{xx}}{\partial t} - (y^{-2} - 2) \frac{\partial \tau_{yy}}{\partial t} \right)
$$

$$
\frac{\partial v_y}{\partial x} = \frac{\partial \tau_{xy}}{\partial t} - \frac{\partial v_x}{\partial y},
$$

Hence,

$$
v^A_x(x + \delta x, (n + 1)\delta t) = v_x(x + \delta x, (n + 1)\delta t) - v_x(x + \delta x, (n + 1)\delta t)
$$

$$
\frac{\delta x}{2(y^2 - 1)} \left( y^{-2} \left[ \tau^H_{xx}(x, (n + \frac{3}{2})\delta t) - \tau^H_{xx}(x, (n + \frac{1}{2})\delta t) \right] \right)
$$

$$
- \left( y^{-2} - 2 \right) \left[ \tau^H_{yy}(x, (n + \frac{3}{2})\delta t) - \tau^H_{yy}(x, (n + \frac{1}{2})\delta t) \right]
$$

and

$$
v^A_y(x + \delta x, (n + 1)\delta t) = v_y(x + \delta x, (n + 1)\delta t) - v_y(x + \delta x, (n + 1)\delta t)
$$

$$
\frac{\delta x}{2\delta y} \left[ \tau^H_{xy}(x, (n + \frac{3}{2})\delta t) - \tau^H_{xy}(x, (n + \frac{1}{2})\delta t) \right]
$$

$$
- \left[ v_x(x + \delta y, (n + 1)\delta t) - v_x(x + \delta y, (n + 1)\delta t) \right].
$$

These added velocities are then equated to additional applied forces, as before, recorded for later use and the values of the velocity components exterior to the $x=$constant parts of the boundary are updated by adding $v^A(x + \delta x, (n + 1)\delta t)$ there. The grid points exterior to the $y=$constant parts of the boundary are then considered in a similar way, but making use of the $\partial / \partial y$ derivatives. Hence time series are obtained for the force components to be applied to the boundary points and points one grid point outside the boundary to reproduce the velocities and stresses on the boundary and the field inside the boundary of the incident excitation. These forces are then also available to be input into any other numerical model for the scattering region, where they will also reproduce the incident field in the presence of a scatterer and will not result in reflections at the artificial boundary. Thus measuring the values on the boundary from this model and subtracting the known values for incident stresses and velocities there directly gives the scattered field there which is needed for a further application of the method to calculate the scattered field at a distance.

**NUMERICAL EXAMPLES**

The original hybrid model has previously been shown, [1,2], to be able to predict the time series for the values of stress and velocity of the incident wave on the boundary of region 2. Here we show a numerical example demonstrating the modified forcing part of the method.

**Finite Difference Example**

To test out the concept a finite difference model for elastic material excited by a point source which generates a pulse consisting of both longitudinal and shear waves is run in the time domain. The time series for values of stress and velocity on the boundary of region 2 are extracted and recorded. These are then used as input to the new part of the method to generate the forces which would need to be applied on and just outside of the region 2 boundary, in order to excite the same incidence wave inside Region 2, if only that small area, surrounded by PML, is used for the calculation.

An animation showing the pulse propagating across Region 2, both from the whole area calculation and the small area calculation using the applied forces calculated as described above will be presented. A still, taken from that animation while the pulse is crossing Region 2 is included here as figure 4. The plot shows the velocity in the x-direction. At the time shown there have already been 332 time steps. There is excellent agreement between the methods at each of the time steps as the pulse crosses the region. Plots of other variables are as expected and also exhibit good agreement.
Finite Element Example
As the analysis and coding to generate the values of the applied forces has been implemented using finite difference methods, a more demanding test of the concept is to use a different calculation method in both the whole area calculation and the two small area calculations and to link them, as intended, using a ‘black-box’ hybrid method to generate the forces to be applied. Thus, we again set up a source box (Region 1) and a defect box (Region 2). The system is excited by a plane wave pulse from the source box. With no defect present in the Region 2 box, comparing the values for the incident wave on the boundaries of the defect box shows good agreement between the whole area calculation and the hybrid calculation with applied forcing on the region 2 boundary, as shown in Figure 5. In addition, the method has been applied when the defect box does contain a defect – in this case a side-drilled hole. Figure 6 shows a comparison of the values on the bottom edge and the left hand edge of Region 2 for this case, as a function of node-number along the edge (y-axis) and time (x-axis) for both the full FE calculation and using the hybrid method with applied forcing at the edges of the defect box. Again there is good agreement between the full, whole area, FE calculation and the hybrid result.

Figure 4 Finite Difference Comparison of Whole Region Calculation and Forcing Region 2 with Applied Forces from Hybrid Calculation
The hybrid method has correctly reproduced the incident field, and this applied force modification also allows the field scattered by the defect to pass through the numerical boundary without introducing unwanted numerical scattering there.

**CONCLUDING REMARKS**

It has been demonstrated that the concept of linking two smaller calculation domains by this hybrid model to calculate the forces which need to be applied to the second domain in order to reproduce the incident wave there is feasible. This method of forcing the second region allows the field scattered by the defect to pass through the computational boundary without introducing numerical scattering there. Hence, this forcing technique permits the size of the second domain to be reduced compared to the version of the hybrid model with line forcing in front of region 2. As the values of the incident field on the domain boundary are available from the hybrid method it is straightforward to extract the scattered field there from the total field obtained from the numerical calculation in Region 2. This is then directly available for a further application of the hybrid method to obtain the scattered field elsewhere, at a distant receiver, for example, or for scattering at a different defect.
The model has also been extended to three-dimensions, in which case there are more variables to include, and more corners and edges which need special treatment. Further development of both the 2D and 3D versions of the codes is on-going.

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**REFERENCES**

