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Wavelet-based Local Tomography

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Abstract

In this paper we develop an algorithm to reconstruct the wavelet coefficients of an image from the Radon transform data. The proposed method uses the properties of wavelets to localize the Radon transform and can be used to reconstruct a local region of the cross-section of a body using almost completely local data which significantly reduces the amount of exposure and computations in X-ray tomography

Keywords: Global tomography, Radon transform wavelets, Local tomography

1. Introduction

The Radon transform was first introduced by J. Radon in 1917. Little computational attention was given to it until the advent of computers enabled the fast evaluation of Fourier transforms and their corresponding convolution. The Radon transform is now a mainstay of medical imaging as well as many other remote imaging sciences.

One problem with the Radon transform is that in 2-D, where most medical imaging is conducted, the inversion formula is globally dependent upon the line integrals of the object function f (x, y) [5]. In many situations a physician may only be interested in images of a very local area of the body. One would prefer to expose only that local portion of the patient's body to whatever radiation is being used. However, the non-locality of the Radon transform forces one to expose to radiation to all the 2-D slices of the body which intersect with the region of interest.

There have been a number of attempts to alleviate this problem. One approach is to do 3-D tomography, but this requires integrals over full 2-D hyper planes, which we are trying to avoid. K.T. Smith and F. Keinert [6] was introduced Lambda tomography which does not attempt to reconstruct the function f (x, y) itself but instead produces the related function $L f = \Lambda f + \mu\Lambda^{-1} f$. This has an advantage that the reconstruction is strictly local in the sense that computations of L f (x, y) require only integrals over lines passing arbitrarilyclose to (x, y). But it requires knowledge of what kind of useful information about f (x, y) is retained in L f (x, y). And also this approach is well adapted for edge detection, but is not designed to recover the original density of the image.

Wavelets [3] have received much interest by engineering community in the past decade, because it has an important property of localization nature in both time and frequency. In [4], Olson and DeStefano implement a direct reconstruction algorithm in which the 1-D wavelet transform of the projection data is computed for each angle. Delaney and Bresler [7] compute the 2-D separable wavelet transform of the image directly from the projection data. Both algorithms take advantage of the observation, made in [8], that Hilbert transform of a function with many vanishing movements has rapid decay. In fact, the Hilbert transform of a compactly supported wavelet with sufficiently many vanishing movements has essentially the same support as the wavelet itself. Thus in both algorithms, the high resolution parts of the image are obtained locally, and the low resolution parts are obtained globally.

It has been observed that, in some cases, the Hilbert transform of a compactly supported scaling function also has essentially the same support as the scaling function itself. We take advantage of this observation to reconstruct the low resolution parts of the image as well as the high resolution parts using almost local data plus a small margin for the support of the filters. This gives substantial savings in exposure of computations over the methods in [4] and [7].

The proposed method calculates the wavelet coefficients of the reconstructed image with the same complexity as the conventional filtered back projection method. The wavelet coefficients are obtained directly from the projection data, which saves the computations required to obtain the wavelet coefficients from the reconstructed image.

The main features of our algorithm are: (1) Reduced exposure compared to previous algorithms. (2) Computatio-nally more efficient than others because we use smaller exposure lengths. (3) Uniform exposure at all angles which allows easier implementation in hard ware. (4) Ability to reconstruct off-center or even multiple regions of interest, as well as centered reconstruction.

In Section-II describes the fundamentals of Tomography and filtered back projection algorithm briefly. The non-locality nature of the Radon transform also explained in this section. Section-III describes the fundamentals of wavelets, multi resolution analysis and synthesis and reconstruction formulas using wavelets. In section-IV describes the local reconstruction technique and algorithm. Section-V is devoted for computer simulations and results. Section-VI gives the conclusions, and finally references in Section-VII.

2. Fundamentals of Tomography

2.1 Radon Transform

In Computerized Tomography (CT), a cross section of the human body or an in destructive object is scanned by a non-diffracting thin X-ray beam whose intensity loss is recorded by a set of detectors. The Radon Transform is a mathematical tool which is used to describe the recorded intensity losses as averages of the tissue density functions over hyper planes which, in dimension two, are lines [1].

Let f(x, y) be a two-dimensional density function of the object which is usually called the object function, or the image function restricted to a disc of radius one. Then the Radon Transform of the function f(x, y) represented by

$$p_\theta(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\ \delta(x\cos\theta + y\sin\theta)\,dx\,dy$$

where δ is the one-dimensional Dirac delta-function and $t = x\cos\theta + y\sin\theta$ is the equation of a line along which the projection has been measured.

2.2 Filtered Back Projection

In order to reconstruct the function f(x, y) from the projection data we use the filtered back projection method. If projections are known at enough angles, the object function can be reconstructed by using the formula

$$f(x,y)=\frac{1}{(2\pi)^2}\iint S_\theta(\omega)e^{j\omega(x\cos\theta+y\sin\theta)}\,|\,\omega\,|\,d\omega d\theta$$

Where $S_\theta(\omega)$ is Fourier transform of $P_\theta(t)$. This algorithm can be implemented in two steps.

- The first step is the filtering operation

  $q_\theta(t)=H\frac{\partial}{\partial t}P_\theta(t)$

  which is the IFFT of $S_\theta(\omega)|\omega|$. (1)

- The second step is the back projection

  $f(x,y)=\int_0^\pi Q_\theta(x\,cos\theta+y\,sin\theta)d\theta$

Where H is the Hilbert transform and $\frac{\partial}{\partial t}$ represents the partial differentiation.

Insertion of an appropriate band limited window into the filtering yields the filtered back projection formula

$$f(x,y)=\frac{1}{(2\pi)^2}\iint S_\theta(\omega)e^{j\omega(x\cos\theta+y\sin\theta)}(|\,\omega\,|\,H(\omega))d\omega d\theta$$
$$=\frac{1}{(2\pi)^2}\int_0^\pi P_\theta(t)*\mathrm{IFFT}(|\,\omega\,|\,H(\omega))$$

(2)

The window H(ω) must be chosen to meet two criteria. Eirst the window should be chosen to agree with the essential band limit of ω, so that the resultant will be a good approximation to f(x, y). Second, the window also represents a mollification of the inversion of the Radon transform, which is an unbounded operator without the addition of the window [6].

2.3 Nonlocality

The problem with the reconstruction formula (2) is that the inverse Fourier transform of | ω | H(ω) will not be locally supported. This stems from the fact that | ω | is not differentiable at the origin. The non locality of this inverse Fourier transform implies that local calculation of the convolution in (2) will require global values of the Radon transform. The non differentiability at the origin can not be significantly altered by the window, without harming the structure of the image.

From (1), the differentiation is local, but Hilbert transform is not, since it imposes a discontinuity upon the Fourier trans-form of any function whose average value is not zero, and discontinuities on the higher derivatives which are not zero at the origin. The imposition of these discontinuities at the origin in the frequency domain will therefore spread the support of functions in the time domain. For this reason, a local basis will not remain local after filtering.

For the reasons outlined above, we would like a basis of functions which are essentially compactly supported, and which possess several zero moments. This second condition will ensure that the basis functions remain essentially compactly supported after the filtering process, and so will allow a reconstruction from localized data.

Wavelets are generally constructed with as many zero moments as possible, given other constraints such as locality and smoothness [2] and [3]. Thus, the local properties of the high resolution components of a wavelet transform will remain local after the filtering in the reconstruction of the image from the Radon transform

3. Fundamentals of Wavelets

3.1 Wavelet Transform

The wavelet transform is a tool which is used in many different areas such as signal and image processing, numerical analysis and tomography. Here, we assume that the reader is already some what familiar with the theory of wavelets; hence, we will only highlight some of its results relevant to this work.

Wavelets are functions derived by shifts in position and scale from a single function ψ(t) called the 'basic (mother) wavelet'. It is possible to construct orthonormal wavelet basis, from a so-called scaling function φ(t), which is the basic wavelet and it can be associated in a unique way. The function φ(t) generates a so-called multi resolution analysis, represented by the basis functions

$$\varphi_{j,k}(t) = 2^{j/2}\varphi(2^j t - k), \quad j, k \in Z \quad \text{and}$$

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k), \quad j, k \in Z,$$

where Z is a set of integers. The refinement equations of scaling and wavelet functions are given by

$$\varphi(t) = 2^{½}\textstyle\sum_n h[n]\,\varphi(2t - n) \text{ and}$$

$$\psi(t) = 2^{½}\textstyle\sum_n g[n]\,\psi(2t - n),$$

where {h[n]} and {g[n]} are a pair of digital low-pass and high-pass quadrature-mirror filters that are related through

$g[n] = (-1)^{1-n} h[1-n]$.

The relationship between the wavelet and scaling function can also be represented in frequency domain as

$$\Phi(\omega) = H(\omega/2)\Phi(\omega/2) \text{ and } \Psi(\omega) = G(\omega/2)\Psi(\omega/2),$$

where we have used Φ(ω) and Ψ(ω) are the Fourier transforms of φ(t) and ψ(t): H(ω/2) and G(ω/2) are the Fourier transforms of the quadrature mirror filters and are both periodic functions with period 2π.

The dilations and translations of φ(t) induce a multi resolution analysis (MRA) of $\mathbf{L}^2(\mathbf{R})$, i.e., a nested chain of closed subspaces $\{\mathbf{V}_m\}$ whose union is dense in $\mathbf{L}^2(\mathbf{R})$. Here, $\mathbf{V}_j$ is the subspace spanned by

$$2^{j/2}\varphi(2^{j/2} t - k), \; -\infty \le k \le \infty.$$

3.2 Wavelet Reconstruction

In this section we will present an algorithm which can be used to obtain the wavelet coefficients of a function on $\mathbf{R}^2$ from its Radon Transform data. Given a real valued square integral function 'g' on $\mathbf{R}^2$ and let f(x) given in $\mathbf{R}^2$, then the Wavelet transform of the function f(x) can be represented by

$$W(a,b) = \int_{-\infty}^{\infty} g_{a,b}((t-b)/a))dx = f(-b) * g_{a,0}(-b)$$

From this definition, the wavelet transform of the Filtered Back Projection formula can be written

$$\text{W.Tof } f(x,y) = W_f(x,y) = \frac{1}{(2\pi)^2}\int_0^{\pi} P_\theta(t')*H\frac{\partial}{\partial t}P_g(t')d\theta$$

Where $P_g(t')$ is the Radon Transform of $g_{a,0}(-b)$ and $t' = a\,x\cos\theta + a\,y\sin\theta$

In the discrete form the filtered back projection can be written as

$$W_f(n_1,n_2) = \frac{1}{(2\pi)^2}\int_0^{\pi} P_g(n)*H\frac{\partial}{\partial t}P_g(n)d\theta,$$

where $n = 2^{-j} n_1 \cos\theta + 2^{-j} n_2 \sin\theta$. This algorithm can be implemented in two step process as below.

Step1: Filtering

$$q_{2^j,\theta}(t) = P_\theta(2^{-j}t) * H\frac{\partial}{\partial t}P_g(2^{-j}t)$$

Step2: The back projection

$$W_f(n_1,n_2) = \frac{1}{(2\pi)^2}\int_0^{\pi} Q_{2^j,\theta}(n_1\cos\theta + n_2\sin\theta)d\theta$$

The filtering part can be implemented in Fourier domain as

$$Q_{2^j,\theta}(\omega) = S_\theta(\omega)\,|\,\omega\,|\,G_{2^j}(\omega\cos\theta,\omega\sin\theta)W(\omega)$$

, where G_{2^j} (ω cosθ, ω sinθ) is the Fourier Transform of $g_{a,0}(.)$, $S_\theta(\omega)$ is the Fourier Transform of $P_\theta(.)$, $Q_{2^j,\theta}(\omega)$ is the Fourier Transform of $q_{2^j,\theta}(.)$, and W(ω) is a smoothing window. Therefore the wavelet based filtered back projection can be implemented using the same algorithm as the conventional filtered back projection method while the ramp filter |ω| is replaced by wavelet ramp filter $|\,\omega\,|\,G_{2^j}(\omega\cos\theta,\omega\sin\theta)$.

4. Multi Resolution Resolution Reconstruction

If the wavelet basis is separable, and defining φ(x,y) = φ(x)φ(y), $\psi^1(x,y)$ = φ(x)ψ(y) $\psi^2(x,y)$ = ψ(x)φ(y), and $\psi^3(x,y)$ = ψ(x) ψ(y). Then the approximation co-efficient are obtained by

$$A_{2^j} f[m,n] = 2^{j/2}\int_0^\pi [(H\partial R_\theta \varphi_{2^j} * P_\theta)((2^{-j}m)\cos\theta + (2^{-j}n)\sin\theta)]d\theta$$

These co-efficients can be calculated using standard filtered backprojection method, while the filtering part in the Fourier domain is given by

$$Q_{A_{2^j},\theta}(\omega) = S_\theta(\omega)\,|\,\omega\,|\,\Phi_{2^j}(\omega\cos\theta,\omega\sin\theta)W(\omega)$$

where Φ_{2^j}(ω cos θ, ω sin θ) = Φ_{2^j}(ω cos θ) Φ_{2^j}(ω sin θ). The detail co-efficients can be found in a similar way as

$$D^1_{2^j} f[m,n] = 2^{j/2}\int_0^\pi [(H\tfrac{\partial}{\partial t}P_\theta \psi^1 2^j * P_\theta)((2^{-j}m)\cos\theta + (2^{-j}n)\sin\theta)]d\theta$$

ℓ=1,2and 3. This means that the wavelet and scaling co-efficients of the image can be obtained by filtered back projection method while the ramp filter is replaced by scaling and wavelet ramp filters, given below.

$$H_\theta^A = |\,\omega\,|\,\Phi_{2^j}(\omega\cos\theta,\omega\sin\theta) = |\,\omega\,|\,\Phi_{2^j}(\omega\cos\theta)\Phi_{2^j}(\omega\sin\theta)$$

$$H_\theta^{D^1} = |\,\omega\,|\,\Psi_{2^j}(\omega\cos\theta,\omega\sin\theta) = |\,\omega\,|\,\Phi_{2^j}(\omega\cos\theta)\psi_{2^j}(\omega\sin\theta)$$

$$H_\theta^{D^2} = |\,\omega\,|\,\Psi_{2^j}(\omega\cos\theta,\omega\sin\theta) = |\,\omega\,|\,\psi_{2^j}(\omega\cos\theta)\Phi_{2^j}(\omega\sin\theta)$$

$$H_\theta^{D^3} = |\,\omega\,|\,\Psi_{2^j}(\omega\cos\theta,\omega\sin\theta) = |\,\omega\,|\,\psi_{2^j}(\omega\cos\theta)\psi_{2^j}(\omega\sin\theta)$$

To reconstruct the image from these coefficients, the discrete approximation at resolution 2^{j+1} can obtained by combining the detail and approximation at resolution 2^j, [3],[9]

$$\begin{aligned} A^d_{2^{j+1}}(m,n) = {} & 2\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h(m-2k)h(n-2l)A^d_{2^j}(k,l) \\ & + 2\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h(m-2k)g(n-2l)D^d_{2^j,1}(k,l) \\ & + 2\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} g(m-2k)h(n-2l)D^d_{2^j,2}(k,l) \\ & + 2\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} g(m-2k)g(n-2l)D^d_{2^j,3}(k,l) \end{aligned}$$

5. Local Reconstruction

It can be shown that if the wavelet function has sufficient number of zero moments, the Hilbert transform does not essentially increase the support of basis. We have noted that this is the case even for scaling functions when we use wavelet basis with sufficient number of vanishing moments. Fig.1 shows the Daubechies biorthogonal wavelet and scaling function and also the Ramp filtered version of these functions. Therefore the Ramp filtered scaling function does not spread either for the wavelet basis.

Using this fact we have devised an algorithm to reconstruct local regions using only local data plus a small margin of the support of the reconstruction filters. In local reconstruction artifacts are common close to the boundary of the ROI. In order to avoid these artifacts, we have extended the $P_\theta(t)$ continuously to be constant on the missing projections. The algorithm proposed is described briefly in the following steps.

Step 1: Obtain the Sinogram of the Head Phantom.

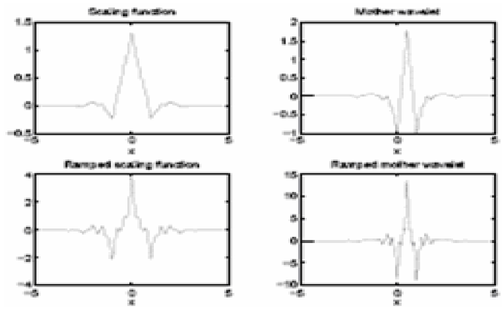

Fig.1: Top, the scaling and wavelet basis in Daubechies biorthogonal wavelet basis. Bottom, the ramp filtered version of the scaling and Wavelet functions

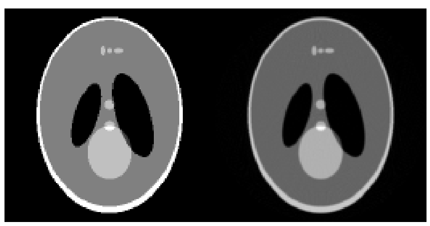

Fig.2: Left: the shepp-logan head Phantom Right: The reconstruction using FBP

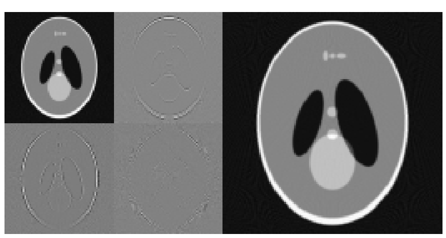

Fig. 3: Left: Wavelet reconstruction Right: Reconstruction from the wavelet coefficients

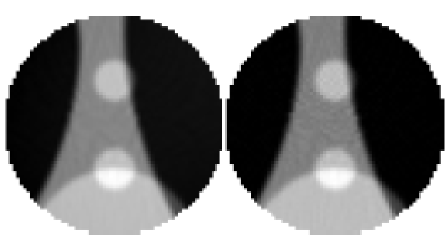

Fig.4: Blowup of the centered region of interest Left: Reconstruction using local data Right: Reconstruction using standard FBP method using global data

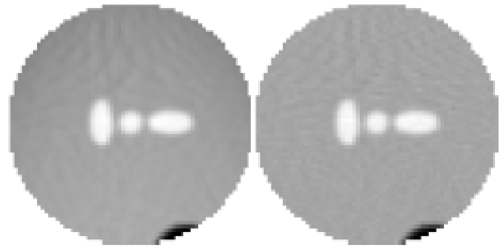

Fig. 5: Blowup of the centered region of interest Left: Reconstruction using wavelet method using local data. Right: Reconstruction using standard FBP using global data

Step 2: Perform the 2-D multi resolution analysis on these projections, to get approximation and detail coefficients at appropriate level.

Step 3: Select the required region of interest in each approximation and detail coefficients obtained in step 2, and reconstruction the region of interest projections.

Step 4: Perform filtered back projection to get the required region of interest image.

6. Computer Simulations and Results

We have obtained the one level approximation and detail coefficients of the 256x256 pixel image of the Shepp-Logan head phantom using global data. (Fig.4). In this decomposition we used Daubechies' biorthogonal filters (Fig 2). The quality of the reconstructed image is the same as with the filtered back projection method (Fig 3). Figs. 5 and 6 show two examples in which two regions of interest are reconstructed using the local reconstruction method proposed in this paper. In Fig .5 and 6 the blow up of the ROI using both local reconstruction and standard filtered back projection using global data is shown for comparison. We have reconstructed the off-center disk of radius 32 pixels located 80 pixels from the center of the image. (Fig 6).in both Fig 5&6 the projections are collected from a disk of radius 44 pixels, therefore the amount of exposure is 17% of the conventional filtered back projection method.

7. Conclusions

In this paper we have explained the non locality nature of the Radon transform and how the local data can be extracted using wavelet transform of the projection data. The algorithm presented here uses the properties of wavelets to localize the Radon transform and can be used to reconstruct a local region of a cross section of a body with significantly reduced exposure in X-ray tomography. The results presented here shows, the ability to reconstruct off-center or even multiple regions of interest, as well as centered reconstruction more efficiently.

8. References

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