1-Dimensional Thermal Wave: Theory, Experiment and Simulation

Krishnendu Chatterjee and Suneet Tuli
Centre for Applied Research in Electronics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi 110016
E-mail: krishnendu.chatterjee@gmail.com, suneet.tuli@gmail.com

Abstract
Thermal imaging or Thermography, as it is often called, has gained significant importance in the field of Non-Destructive Testing. Thermography is classified into two broad categories: 1) where the target is not externally heated — known as Passive Thermography and 2) where the target is externally heated — known as Active Thermography. The major application of the passive thermography is preventive maintenance, e.g. maintenance of electrical wiring and break system of vehicles, while the active thermography is primarily used for subsurface defect detection inside materials. With time many heating schemes were proposed for active thermography. Pulsed Heating, periodic heating with constant frequency (Lock-in) and periodic heating with varying frequency (FMTWI), etc. to name a few. The duration of the pulsed heating technique is short — in the order of a 1 minute. However this technique imposes a very hard restriction on the maximum magnitude of the pulse. The periodic heating distributes the incident energy over longer time removing any such restriction. However it demands better understanding of the heat flow inside the material specially in the context of thermal waves.

This paper primarily focuses on the understanding of the nature of the thermal wave inside a material. The propagation of the thermal wave through a metal (Aluminium) is modelled, simulated and the value of its thermal diffusivity (m) is calculated and experimentally measured. It is then compared with that obtained from the theory of one dimensional heat flow in a semi-infinite material. A model of radiation and convection loss is also proposed and implemented in the simulation.

1. Introduction
If a body is heated, heat flows in all 3 directions (X, Y and Z) inside the body obeying the laws of thermodynamics. However, if the body is made markedly long in one dimension (say X) and relatively thin in the others (Y and Z), then the flow of the heat can be thought to be approximately confined to only one dimension.

Based on the analogy between the thermal and the electrical quantities [1], such a body (a bar) can be modelled using a series of resistors and capacitors. Quantitatively this 1-D RC chain is expected to behave like a low pass filter when under external periodic excitation, i.e. the resulting signal (wave) inside the network suffers phase shift (lags) between two successive nodes and gradually gets attenuated as one goes away from the source. Thus if the bar is periodically heated from one end, an attenuated (thermal) wave is expected to propagate along its length.

2. Theory of 1-Dimensional thermal wave
When a surface is heated periodically, a highly attenuated and dispersive wave propagates through the material whose equation for a semi-infinite medium can be written as [2],

\[ T(x,t) = T_0 e^{-\mu t} \cos \left( \frac{\omega t - \frac{2\pi x}{\lambda}}{\lambda} \right) \]

where \( \mu \) is the thermal diffusion length, expressed by

\[ \mu = \sqrt{\frac{2k}{\rho c}} = \sqrt{\frac{2\alpha}{\omega}} \]  

(2)

with thermal conductivity \( k \), density \( \rho \), specific heat \( c \), modulation frequency \( \omega \) \( [= 2\pi f \text{ (rad s}^{-1})] \), frequency in Hertz, thermal diffusivity \( \alpha \) and thermal wavelength \( \lambda \), defined as

\[ \lambda = 2\pi\mu \]  

(3)

In equation 1, notice that the phase of the thermal wave \( \phi(t,x) \) at a given \( t \) (say \( \tau \)) is a linear function of the depth \( x \).

\[ \phi(x) = \left( \omega \tau - \frac{2\pi x}{\lambda} \right) = \left( \omega \tau - \frac{x}{\mu} \right) \]

(4)

Thus the plot of \( \phi(x) \) vs \( x \) at a given time yields a straight line with a slope of \(-1/\mu\).

Similarly the amplitude of the thermal wave \( A \) is also a function of \( x \).
\[ A(x) = T_0 e^{-x/\mu} \] (5)

Taking log of both the sides
\[ \ln A(x) = \ln(T_0 e^{-x/\mu}) \]
\[ \Rightarrow \ln A(x) = \ln T_0 - x/\mu \] (6)

Thus, once again the plot of \( \ln A(x) \) vs \( x \) will be a straight line with slope of \(-1/\mu\).

For Aluminium, the value of \( \mu \) at 0.01Hz can be calculated from equation 2 using the values [2] in Table 1:

Table 1: Physical parameters of Aluminium

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal conductivity ( k )</td>
<td>230 (W m(^{-1}) oC(^{-1}))</td>
</tr>
<tr>
<td>Density</td>
<td>2700 (Kg m(^{-3}))</td>
</tr>
<tr>
<td>Specific heat ( s )</td>
<td>880 (J Kg(^{-1}) oC(^{-1}))</td>
</tr>
</tbody>
</table>

\[ \mu_{\text{Aluminium}} = \frac{2 \times 230}{2 \pi \times 0.01 \times 2700 \times 880} \approx 55.5\text{mm} \] (7)

Next, in Section 3, the aforesaid value of \( \mu \) is compared with that calculated from the experimentally obtained plots of \( \phi(x) \) vs \( x \) and \( \ln A(x) \) vs \( x \).

Consecutively, in Section 3, the same is done using the plots obtained from the simulation.

These data combined cross-verify the theory, experiment and the modelling of the 1D thermal wave.

3. Experimental verification

The experimental setup to study the thermal waves inside the material is shown in Fig. 1. A hole of diameter 10mm and depth 32mm was made at one end of the bar and a cartridge heater of identical dimension was inserted into the hole.

To attain steady state heat flow, the bar was subjected to periodic heating by the cartridge heater for one hour. The maximum heater power and the heating frequency were 50W and 0.01Hz respectively. Figure 2 shows the time variation of temperature for various sections of the bar under steady state, recorded using an infra-red camera (CEDIP MWIR).

3.1 Finding the amplitude and the phase for a given pixel

A movie with the special resolution of 320 × 240 was taken at 4Hz frame-rate for 1000 seconds. Since the time period of the heating is 100 seconds, 10 complete heating cycles were recorded in the movie.

Plot of temperature \( y \) vs time \( t \) graph for a given pixel has three components — a DC offset, an AC variation and noise, i.e.

\[ y(t) = y_{|dc} + A_0 \sin(\omega t + \phi) + y_{|\text{noise}}(t) \] (8)

There can be a slow variation observed in \( y_{|dc} \) because of the fluctuation of ambient temperature and the asymptotic nature of stabilisation. To get rid of this, the entire signal is divided into 10 equal parts \( (y_1, y_2, y_3, ..., y_{10}) \) with 1 complete cycle in each block (100 points). Then

\[ y_i(t) = y_{|dc} + A_0 \sin(\omega t + \phi) + y_{|\text{noise}}(t) \] (9)

Superposing the \( y_i \)'s and normalising produces an average signal \( Y(t) \) with 100 data points.

\[ Y(t) = Y_{|dc} + A_0 \sin(\omega t + \phi) + Y_{|\text{noise}}(t) \] (10)

Since the noise is random in nature, \( Y_{|\text{noise}}(t) \) will be substantially smaller than \( y_{|\text{noise}}(t) \). To find out \( Y_{|dc} \), \( Y(t) \) can be integrated over one cycle (\( \tau \)) and then normalised.

\[ Y_{|dc} = \frac{1}{\tau} \int_0^\tau Y(t)dt \] (11)
Thus the AC component of the signal can be written as

\[ Y_{ac}(t) = Y(t) - Y_{dc} = A_0 \sin(\omega t + \phi) \] (12)

\( A_0 \) can be found out by integrating the square of \( Y_{ac}(t) \) over one cycle \( (\tau) \).

\[
\frac{1}{2} \int |Y_{ac}(t)|^2 dt = \frac{1}{2} \int (A_0 \sin(\omega t + \phi))^2 dt
\]

\[ A_0 = \sqrt{\frac{2}{\tau} \int |Y_{ac}(t)|^2 dt} \] (13)

Once \( A_0 \) is known, \( \phi \) can be found out by best fit sine function using the least square method.

### 3.2 Amplitude and Phase vs Pixel

The movie data, processed as outlined in Section 3.1 was then used to bring out both amplitude and phase variations as a function of pixel denoting propagation direction. Figure 3 shows the amplitude vs pixel and phase vs pixel plot for the bar.

As one goes away from the source, the signal strength is expected to drop down, making the amplitude and the phase measurements more and more erroneous. This can be seen in the above plots where the data from 150th pixel onward become progressively noisy.

Straight line fits in between the 25th to 100th pixel using least square method for both the log amplitude vs pixel and the phase vs pixel plot produce slopes of \(-0.024915/\text{pixel}\) and \(-0.023887/\text{pixel}\) respectively. In order to change the unit to \(/\text{meter}\), the following pixel-to-distance mapping was done.

\[
\begin{align*}
\text{From log Amplitude vs Pixel plot} & \quad \text{From Phase vs Pixel plot} \\
-0.024915/\text{pixel} & \quad -0.023887/\text{pixel} \\
\times 1.2561/\text{mm} & \quad \times 1.2561/\text{mm} \\
= -0.0198352/\text{mm} & \quad = -0.0190168/\text{mm} \\
= -19.83/\text{meter} & \quad = -19.02/\text{meter}
\end{align*}
\]

In the recorded image, which is in 1:1 scale, 41 vertical pixels map to 51.5 mm distance (line A and B in Fig. 2), i.e. 1 pixel \(= 1.2561 \text{ mm} \). So the slopes of the fitted lines in terms of /meter can be written as.

The empirical values of the thermal diffusion length (1 slope) calculated from the above log amplitude vs pixel and the phase vs pixel plot are 50 mm and 52.5 mm respectively.

The theoretical value of the thermal diffusion length as calculated from equation 7 in section 2 was 55.5 mm.

### 3.3 Error in Slope Measurement

Let \( s \) be the slope of the fitted line in /pixel unit and \( p \) pixels map to \( l \) meter distance in the image. The slope \( S \) in /meter unit will be

\[ S/\text{meter} = (s/\text{pixel}) \times \frac{p}{l} \text{ pixel} \]

Taking natural log of both sides and differentiating, contribution to the variations in slope, \( dS \), then are

\[ dS = S \left( \frac{ds}{s} + \frac{dp}{p} + \frac{dl}{l} \right) \]

The maximum error in \( S \) \( S_{\text{error}}^{\text{max}} \) happens when all the right hand terms add up. Thus

\[ S_{\text{error}}^{\text{max}} = S \left( \frac{ds}{s} + \frac{dp}{p} + \frac{dl}{l} \right) \] (14)

The \( \chi^2 \) values of the best fitted straight lines are > 0.99 indicating that the contribution of \( ds/s \) term is negligible. The maximum error in the vertical pixel count (\( dp \)) can be 1 pixel over 41 pixels (\( p \)) while that in length measurement (\( d'l' \)) can be 1 mm (variation in the width of the bar) over 51.5 mm (\( l \)).

\[ S_{\text{error}}^{\text{max}} \approx 19.4 \left( 0 + \frac{11}{41} \right) /\text{meter} \]

\[ \Rightarrow S_{\text{error}}^{\text{max}} \approx 0.8/\text{meter} \]

Fig. 3 : Log of the amplitude and Phase vs pixel plot for the bar @ 0.01Hz excitation
Thus, the average value of the experimentally determined slope from the log(A) vs x and (x) vs x plots is 19.4 ± 0.8/meter.

The diffusion length $\mu$ and its error ($d\mu$) can now be calculated.

$$\mu = \left| \frac{1}{S} \right| = \frac{1}{19.4} \text{ meter } = 51.5 \text{ mm}$$

$$\Rightarrow d\mu = -\mu \times \frac{dS}{S} = -\left( \frac{dS}{S^2} \right)$$

Thus maximum error in $\mu (\mu_{\text{max}}^{\text{error}})$ is

$$\mu^{\text{error}}_{\text{max}} = \frac{0.8}{19^2} \text{ meter } = 0.002 \text{ meter } = 2 \text{ mm}$$

The experimentally observed value of thermal diffusion length $\mu$ therefore is 51.5 ± 2 mm.

References

2. Xavier P.V. Maldague, Theory and practice of infrared technology for nondestructive testing.