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Piecewise semi-analytical modeling of guided wave generation by piezoelectric transducers

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ABSTRACT

Embedded piezoelectric transducers (PZTs) are used in Structural Health Monitoring (SHM) for excitation and sensing of guided waves (GWs) which have the advantage of propagating over long distance in plate-like structures. In this paper, a semi-analytical solution is formulated for representing through-the-thickness GW propagation in isotropic plates. The main novelty of this method is its ability to simulate GW propagation excited by any arbitrary shaped shear stress profile on an infinite isotropic plate, which is applicable to optimization and design of transducers. An analytical model is first developed based on the 2D linear elasticity equations and Fourier transform to obtain the GW field propagation through the thickness of a plate for different types of excitations using the residue theorem for the inverse transform. Then, the arbitrary shear stress generated by a bonded PZT is obtained numerically using the commercial software ANSYS and provided as an input to the analytical model. For this purpose, the shear stress profile is divided into a finite number of segments and equivalent shear stresses are substituted sequentially with the average of upper and lower limit of shear stress on each segment. Using the expanded analytical model, the GW field excited by each segment is obtained individually and the final GW field is obtained by superimposing the GW fields generated by the shear stress applied by each individual segment. Several test cases are presented to validate analytical results with those obtained numerically using the commercial software ANSYS. The high accuracy and the capability of the present method to simulate GW excited by arbitrary PZTs are both demonstrated.

Keywords: Structural Health Monitoring, Guided waves, Piezoelectric transducers, Semi-analytical solution.
INTRODUCTION

Structural health monitoring (SHM) is a technology which has emerged in recent years, dealing with the development and implementation of techniques and systems for monitoring, inspecting and detecting damage in structures [1]. A common SHM solution consists of an integrated network of embedded piezoelectric (PZT) transducers which are used for the excitation of guided waves (GW) [2-3]. GW modes are symmetric (S₀, S₁, S₂, …) and antisymmetric (A₀, A₁, A₂, …) and are sensitive to different types of defects [4]. At any given frequency, PZTs generate at least two basic GW modes, S₀ and A₀ [5-6]. PZT actuators generate GWs by applying shear stress to the surface of the plate, which is highly dependent on its mechanical and geometrical parameters [7-8].

Lin and Yuan [9] analytically simulated PZT actuators, based on Mindlin plate theory, to generate pure A₀-mode. They included both transverse shear and rotary inertia in their analysis and used the assumption of bending moments applied along the tip of the actuator. The interfacial shear stress is modeled based on classical laminate theory. Veidt et al [10] studied flexural waves excited by rectangular transducers in an isotropic plate based on Mindlin plate theory. They used a combined theoretical and experimental approach to study the effect of electromechanical properties on the performance of the transducer. The disadvantage of Mindlin plate theory is that it is limited to low frequencies, below the cut-off frequency at which higher antisymmetric GW modes other than A₀ can appear. Giurgiutiu [11] studied the behavior of surface bonded PZT actuators for GW excitation through surface “pinching” or tangential traction at the actuator edges on the structure surface. He derived three-dimensional (3D) linear elasticity equations by assuming the simplified harmonic “pin-force” excitation model of a PZT actuator. Applying Fourier transform and using residue theorem, he simplified equations into two-dimensional (2D) equations and obtained the displacement field for GW excitation [12]. A similar analysis was presented by Raghavan and Cesnik [13] for a 3D model in which rectangular and circular surface-bonded PZT actuators generate GW in an isotropic plate based on harmonic excitation source and the simplified “pin-force” model. Ende et al. [14-15] presented an analytical solution for GW excitation based on 2D elasticity equations with the help of Fourier transform and residue theorem for inverse transform. To overcome the limitations of using pure analytical solutions, commercially available finite element (FE) codes are often used as powerful tools. FE methods (FEM) [16] are time consuming for models with a huge number of nodes, like the ones with large structures or high frequencies to be considered. Therefore, hybrid numerical-analytical methods are used in some research work in which the interfacial shear stress is obtained by FEM, and the GW displacement field is obtained by analytical methods. As an example, Moulin et al. [17] used a coupled FE–normal modes expansion approach, which allows one to consider either the case of bonded or embedded multi-element transducers into composite materials.

In this paper, an analytical model is first developed for single uniform shear stress excitation based on the two-dimensional (2D) linear elasticity equations and Fourier transform. The GW field propagation is obtained through the thickness for a plate with different types of excitations using the residue theorem for the inverse transform. Interfacial shear stress distribution is obtained numerically from FEM. The final GW field is obtained based on the superposition principal. Several test cases are presented to validate semi-analytical results with those obtained
numerically using the commercial software ANSYS [18] and accuracy of the present method are demonstrated.

**METHODOLOGY**

Consider a 2D infinite isotropic plate with total thickness $2d$ in the lateral ($y$−) direction, and infinite length in the longitudinal ($x$−) direction. Any arbitrary harmonic or transient shear stress distribution, either arbitrary or coming from a PZT actuator, excite the top surface of the plate as shown in Figure 1.a and Figure 1.b, respectively. The radius of excitation zone is $r$, PZT actuator thickness is $t_p$, and direction of GW propagation is along the $x$− direction.

![Primary model of GW excitation](image)

**Fig. 1:** Primary model of GW excitation by (a) an arbitrary shear stress distribution (b) a PZT actuator.

**Arbitrary shear stress:** It is assumed that an arbitrary distribution of shear stress excites the top surface of the plate at the excitation zone as it shown in Figure 1.a. The excitation zones is divided into infinite number of segments $i = 1, 2, ..., N$ as shown in Figure 2, where $b_i$ and $a_i$, denote left and right boundaries of segment $i$, respectively. Then, the shear stress of each segment is replaced with an equivalent uniform shear stress denoted by $\tau^{(\text{sub})}_i$, taken as the average of stresses at the two boundaries of segment $i$ and defined as:

$$\tau^{(\text{sub})}_i = \frac{\tau_i(\omega, a_i) + \tau_{i-1}(\omega, b_i)}{2},$$  \hspace{1cm} (1)$$

where $\tau_{i-1}(\omega, b_i)$ and $\tau_i(\omega, a_i)$ are the shear stresses on the left and right boundaries of segment $i$, and $\omega$ is angular frequency. The higher the number of segments, the higher the accuracy obtained on the shear stress. In other words, decreasing the width of each segment will increase the accuracy of approximation of the equivalent shear stress. However, simulations have indicated that convergence is achieved after a limited number of divisions.

It is assumed that each substituted uniform shear stress induces GWs into the plate individually. In this way, there are $N$ sources for propagated GWs in the plate, as shown in Figure 3. Based on the superposition principle, instead of calculating the excited GW from the preliminarily arbitrary shaped shear stress, the total GW displacement field is obtained by superposition of all individual GWs.
Analytical Solution: In this section, an analytical solution is obtained for single uniform shear stress sources to excite GW in the plate. Based on equation of motion, 2D displacement field $u$ is written as:

$$u = \nabla \varphi + \nabla \times \psi,$$

where $\varphi$ and $\psi$ are called the potentials associated with longitudinal and shear waves, respectively and $\nabla$ denotes the gradient operator. For a time harmonic process with $u(x, y) e^{-i \omega t}$, the potentials $\varphi$ and $\psi$ satisfy the following wave equations:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + p^2 \varphi = 0,$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + q^2 \psi = 0,$$

in which, $p = \omega \sqrt{\rho / (\lambda + 2\mu)}$ and $q = \omega \sqrt{\rho / \mu}$ are the wavenumber for longitudinal and transverse modes, respectively; $\lambda$ and $\mu$ are the Lame constants [11], $\rho$ is density. Applying Fourier transform into equation (3), the general solution in the space domain is obtained:

$$\bar{\varphi} = C_1 \sin(p y) + C_2 \cos(p y),$$

$$\bar{\psi} = C_3 \sin(q y) + C_4 \cos(q y),$$

where unknown coefficients $C_2$ and $C_3$ are associated with symmetric modes, and unknown coefficients $C_1$ and $C_4$ are associated with antisymmetric modes which can be determined based on the shear stress conditions on the surfaces [12]. Substituting the coefficients $C_1$, $C_2$, $C_3$, and $C_4$ into equations (4), and subsequently using the Fourier transform of equation (2), the displacement fields are obtained as below:
\[
\bar{u}_{x,y}(\kappa, y) = \frac{N^A_{x,y}(\kappa, y)}{D^A(\kappa)} \bar{T}^A(\kappa) + \frac{N^S_{x,y}(\kappa, y)}{D^S(\kappa)} \bar{T}^S(\kappa).
\] (5)

with the following notations, where \(\bar{u}_{x,y}(\kappa, y)\) stands for \(\bar{u}_x(\kappa, y), \bar{u}_y(\kappa, y)\):

\[
N^A_x(\kappa, y) = q \left( -\left( \kappa^2 - q^2 \right) \sin(dp) \sin(qy) + 2 \kappa^2 \sin(dp) \sin(qy) \right),
\]
\[
N^S_x(\kappa, y) = q \left( -\left( \kappa^2 - q^2 \right) \cos(dp) \cos(qy) - 2 \kappa^2 \cos(dp) \cos(qy) \right),
\]
\[
N^A_y(\kappa, y) = -i\kappa \left( 2pq \cos(pq) \sin(qy) + (\kappa^2 - q^2) \cos(dp) \sin(qy) \right),
\]
\[
N^S_y(\kappa, y) = -i\kappa \left( 2pq \cos(dp) \sin(qy) + (\kappa^2 - q^2) \cos(dp) \sin(qy) \right),
\]
\[
D^A(\kappa) = \mu \left( 4\kappa^2 pq \cos(dp) \sin(qy) + (\kappa^2 - q^2) \cos(dp) \sin(dp) \right),
\]
\[
D^S(\kappa) = \mu \left( 4\kappa^2 pq \cos(dp) \sin(dp) + (\kappa^2 - q^2) \cos(dp) \sin(dp) \right).
\] (6)

The main goal of developing the current analytical method is for those cases in which there is just one single uniform stress as the only source of excitation as shown in Figure 3. Therefore, shear stress distribution can be a set of arbitrarily uniform shear stresses without symmetrical constraint consideration respect to \(y-\)axis. Taking as an example segment 1, a single uniform shear stress can be defined based on the Heaviside step function as:

\[
\tau(x) = \tau_1 \left( H[x - b_1] - H[x - a_1] \right).
\] (7)

Applying the Fourier Transform to equation (7) results in:

\[
\bar{\tau}(\kappa) = -i\tau_1 \frac{\kappa}{\kappa} \left( -e^{-ik\eta} + e^{ik\eta} \right),
\] (8)

where \(a_1\) and \(b_1\) are the left and right boundaries of the excitation zone when uniform shear stress \(\tau_1\) is the excitation source of segment 1.

Substitution equation (8) into equations (5), and taking inverse Fourier transform of the results obtained, the displacement fields for GW are obtained through the thickness of the plate:

\[
u_{x,y}(x, y) = \int_{-\infty}^{\infty} \bar{u}_{x,y}(\kappa, y) e^{-i\kappa x} d\kappa = \int_{-\infty}^{\infty} \left( \frac{N^A_{x,y}(\kappa, y)}{D^A(\kappa)} \bar{T}^A(\kappa) + \frac{N^S_{x,y}(\kappa, y)}{D^S(\kappa)} \bar{T}^S(\kappa) \right) e^{-i\kappa x} d\kappa.
\] (9)

Solving equation (9) based on residue theorem [14] results in:
\[ u^{(1)}_{x,y}(x,y) = -\frac{\tau_1}{4} \left( \sum_{j=1}^{m} \frac{N^A_{x,y}}{\kappa} (\kappa = \kappa_j, y) e^{i\kappa(x-a)} + \sum_{l=1}^{n} \frac{N^S_{x,y}}{\kappa} (\kappa = \kappa_l, y) e^{i\kappa(x-a)} \right) + \frac{\tau_1}{4} \left( \sum_{j=1}^{m} \frac{N^A_{x,y}}{\kappa} (\kappa = \kappa_j, y) e^{i\kappa(x+b)} + \sum_{l=1}^{n} \frac{N^S_{x,y}}{\kappa} (\kappa = \kappa_l, y) e^{i\kappa(x+b)} \right) \] (10)

At Singularities of function \( D^S(\kappa) \) and \( D^A(\kappa) \), their roots are symmetric and antisymmetric eigenvalues. The eigenvalues are obtained using a contour consisting of semicircles in the upper half of complex \( \kappa \) plane and the real axis as shown in Figure 4. This displacement field comes from the shear stress \( \tau_1^{\text{ave}} \) generated on segment 1.

The GWs which are excited by a number of excitation sources result from the superposition of GWs generated by each source. Therefore, using the superposition principle, the total displacement field is obtained by summation of all generated GWs as:

\[ U^{\text{Total}} = \sum_{i=1}^{N} u^i, \] (11)

where \( u^i \) is displacement field generated by shear stress on segment \( i \).

**Fig. 4:** Contour for evaluating the inverse Fourier transform integral. Only the positive wave numbers are included [12].

**Fig. 5:** 2D FE simulation of surface-bonded PZT actuator on an isotropic plate. Interfacial shear stress is extracted at each node in the excitation zone.

**PZT semi-analytical simulation:** The interfacial shear stress of a PZT actuator is highly sensitive to variation of its parameters (frequency, materials, ratio of the thickness of the PZT to the thickness of the host plate). The classical “pin-force” model [12-13] cannot fully take into account this sensitivity, and despite its main assumption, interfacial shear stress is not concentrated just to areas near to the perimeter of PZT actuators. In the presented method, an accurate interfacial shear stress distribution is obtained via FE analysis. FE analysis which is a part of the current hybrid approach is conducted using the commercial software ANSYS. A 2D surface-bonded PZT is simulated as shown in Figure 5. The transient excitation signal is assumed to be uniformly distributed over the electrode surface of the PZT.
To couple the FEM model to the analytical model, the distance between each two adjoining nodes at excitation zone is chosen as the width of each segment. Therefore, the shear stress distribution is obtained at each interfacial node in time domain. The values for the shear stress obtained from FE at each node correspond to those at boundaries of each segment in the analytical method. The GW displacement fields in equation (10) are written based on applied uniform shear stress. Therefore, using equation (1), the average of stresses between two adjoining nodes is obtained.

The method proposed in this paper can facilitate optimization of PZT transducers. In addition, this approach can be used for both simple and complex structures, exploiting the geometrical flexibility brought by FEM. Although, the dynamics of the actuator is neglected in other similar analytical methods [11-13], obtaining shear stress via FE analysis alleviates this limitation.

RESULTS AND DISCUSSION

This section attempts to demonstrate the performance of the presented approach on piecewise semi-analytically modeling of GW generation by PZT transducers. Three examples are presented for studying sensitivity of GW to excitation through simulations of interfacial shear stress distribution, uniform distributed shear stress simulation, and surface-bonded PZT actuator. Based on the theoretical formulation discussed in the preceding sections, the mathematical procedure is coded under the MATLAB environment [19]. Commercial software ANSYS is used in all examples and the element types plane223 and plate183 are used for PZT and isotropic plate simulation, respectively. The material properties used in examples are listed in Table 1, where $S$, $v$, $\rho$, $e$, and $\varepsilon$ denote stiffness constant, Poisson’s ratio, density, piezoelectric stress coefficient, and electrical permittivity. Materials isotropic 1 and PZT 1 are used in the first example and isotropic 2 and PZT 2 are used in the next two examples.

<table>
<thead>
<tr>
<th></th>
<th>$S_{11}$ (GPa)</th>
<th>$S_{22}$ (GPa)</th>
<th>$S_{12}$ (GPa)</th>
<th>$\nu_{12}$</th>
<th>$\rho$ (kg/m$^3$)</th>
<th>$\varepsilon_{12}$ (C/m$^2$)</th>
<th>$\varepsilon_{22}$ (C/m$^2$)</th>
<th>$\varepsilon_{11}$ (C/Vm)</th>
<th>$\varepsilon_{22}$ (C/Vm)</th>
</tr>
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<tr>
<td>Isotropic 1</td>
<td>52.5</td>
<td>-</td>
<td>-</td>
<td>0.33</td>
<td>2700</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Isotropic 2</td>
<td>161.5</td>
<td>-</td>
<td>-</td>
<td>0.30</td>
<td>7800</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>PZT 1</td>
<td>110.8</td>
<td>120.3</td>
<td>75.0</td>
<td>0.33</td>
<td>7750</td>
<td>-5.3</td>
<td>15.7</td>
<td>919.1</td>
<td>826.6</td>
</tr>
<tr>
<td>PZT 2</td>
<td>139.0</td>
<td>115.0</td>
<td>73.4</td>
<td>0.35</td>
<td>7650</td>
<td>-5.2</td>
<td>15.1</td>
<td>728.5</td>
<td>634.7</td>
</tr>
</tbody>
</table>

Example 1: Interfacial shear stress is highly sensitive to some parameters such as the excitation frequency and ratio of the thickness of the PZT to the thickness of the host plate. A 3D model of GW excitation of surface-bonded PZT is simulated using the ANSYS software with geometrical parameters $d = 0.75\text{mm}$, $r = 0.5\text{mm}$, $t_p / d = 1$.

The excitation frequency is varied as $f_0 = 100$ kHz, 450 kHz, and 570 kHz, to investigate the effect of frequency on interfacial shear stress distribution. As shown in Figure 6, the shear stress distribution is not sensitive to thickness ratio at low frequency (100 kHz). Complexity of shear stress increases as the frequency reaches 450 kHz and 570 kHz. Amplitude of shear stress at inner areas far from the perimeter of PZT is not negligible, because its contribution to GW
excitation is similar than closer to the tip of PZT. This shows that the pin-force model is not valid for all conditions, because it ignores the shear stress at inner interfacial areas.

Figure 6: Interfacial shear stress distribution considering the variation of on thickness ratio and the excitation frequency.

Figure 7: 2D FE simulation of GW excitation by uniform shear stress in absence of PZT.

**Example 2:** In order to study excitation by uniform shear stress, a 2D FEM is developed using ANSYS with parameters \( d = 2.4 \text{ mm}, \ f_0 = 60 \text{ kHz}, \) and \( r = 4.5 \text{ mm} \) as shown in Figure 7. Simulation of uniform shear stress \( \tau_{xy} \) in ANSYS is conducted by applying corresponding force to each node parallel to the surface of the plate. The Fourier transform of the transient excitation signal is calculated and used in equation (10). A transient excitation signal with a maximum voltage of 5V, and 6.5 cycles are applied to the top and bottom faces of the PZT, respectively. The in-plane and out-of-plane displacement fields shown in Figure 8 are obtained at \((x, y) = (600, 2.4) \text{ mm}\), both with analytical and FEM methods and compared with each other. A maximum 4% difference is noted between displacements obtained form FEM and analytical method.

**Example 3:** To demonstrate the accuracy of the proposed semi-analytical method, 2D surface-bonded PZT is simulated using ANSYS, as shown in Figure 5. A transient excitation signal with a maximum voltage of 5V, and 6.5 cycles is applied to the top surface of the PZT. It is assumed that the thickness of the electrodes is negligible in comparison with the thickness of the PZT. The distance between two adjoining nodes is considered as the width of segments. In this FEM, 40 quadratic plain strain elements are used in the excitation zone. The proposed analytical method is written based on uniform applied shear stress within a segment. Two cases are studied to obtain the equivalent shear stress to calculate the GW displacement field analytically: 1) shear stress on the left boundary obtained by FEM is used as the equivalent uniform shear stress, or 2) the average of shear stress at each adjoining nodes obtained by FEM is used as an equivalent.
uniform shear stress, as shown in equation (1). The results are presented for the displacement measured at $(x, y) = (600, 2.4)$ mm. The results, when using shear stress on the left noudary as the equivalent shear stress, are shown in Figure 9.a, where a maximum 18% difference is noted between the displacement fields obtained from FEM and from the analytical method. The results, when using the average of shear stress at adjoining nodes as the equivalent uniform shear stress are shown in Figure 9.b. An excellent agreement is obtained, with a maximum 3.5% difference noted between the displacement fields obtained from FEM and from the analytical method. This agreement validates the appropriateness of using equation (1).

**CONCLUSION**

The interfacial shear stress of a PZT actuator is highly sensitive to variation of its parameters (materials, geometry, frequency and ratio of the thickness of the PZT to the thickness of the host plate). It is proved that the simplified “pin-force” excitation model is not accurate in some cases, since the interfacial shear stress is not concentrated just to areas near to the perimeter of a PZT actuator. A semi-analytical solution is formulated for representing through-the-thickness GW propagating in isotropic plates. An analytical method is developed for those cases in which there is a just a single uniform shear stress source for GW excitation. The excitation area is decomposed into a finite number of segments with arbitrary shaped shear stress. The superposition principle issued to obtain the final GW displacement field generated by all the segments. The advantage of this methodology is that multi-element transducers, like phased-arrays, can be easily simulated with the same approach. Any complicated shear stress distribution between a PZT transducer and a plate can be studied by this hybrid semi-analytical method. The proposed method will be used in future work for the optimization of PZT transducers for selective GW mode generation and sensing.
REFERENCES

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