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Mathematical modeling of the radiated acoustic field of ultrasonic transducers

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Abstract

Ultrasonic inspection techniques are widely used for nondestructive evaluation of materials. Mathematical modeling provides an efficient and reliable method to assist the design of the inspection application and the analysis of its results. In this work we use a method based on the Rayleigh-Sommerfeld integral to express the pressure in a fluid generated by different ultrasonic devices like monolithic transducers and phased arrays. Analogous expressions for the displacement in a solid are used to model the case of the inspection of a solid specimen by means of immersion transducers. In order to match the pressure field in the coupling liquid with the displacements in the solid the pressure and displacement fields are expanded in terms of planar waves. Using then the stationary phase approximation to single out the relevant incidence / refraction directions and considering the corresponding transmission coefficients (for a planar interface) the computation of the field at a given point inside the solid can be reduced to an integral over the transducer surface. This integral is in turn approximated in a fast and accurate way by application of the edge elements method. Several examples are presented, involving both monolithic and phased arrays transducers, and a detailed comparison with available analytical solutions. The code developed is also applied to the computation of the delay laws aimed at concentrating the acoustic energy in a given region of the fluid or the solid for an immersion linear phased array.

1 Ultrasonic Transducers in a Fluid

Ultrasonic transducers are used as transmitters to inject a beam of sound into a target specimen and also as receivers to convert the acoustic response into electrical energy. This section models the sound beam generated by transducers acting as transmitters.

1.1 Immersion Transducer

In immersion testing an ultrasonic transducer radiates a sound beam directly into a fluid. This beam must then cross a fluid-solid boundary to enter the part to be tested.

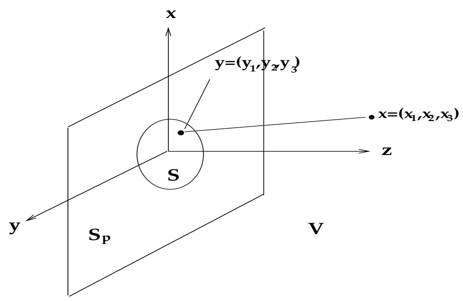

Figure 1: Planar piston transducer in a fluid. The fluid is located in the half-space z $\geq$ 0. The velocity over the planar surface S_p is taken as zero. The normal velocity is taken as a constant over the surface S.

Here is first considered the beam of sound only in the fluid itself. To model the radiation field of a planar transducer, the geometry shown in Figure(1) is considered, where the fluid region is the half space $z \geq 0$ and the boundary of this region is the plane of the transducer. In this model, the motion of the transducer surface acts as a piston and this piston is surrounded by a motionless infinite plane.

1.2 Rayleigh-Sommerfeld Integral

One way to model the planar piston transducer is applying to the half-space $V(z \geq 0)$ in Figure(1) the integral representation theorem(1) for the pressure in the fluid . Assuming the radiated pressure field satisfies the Sommerfeld radiation conditions(1), then the pressure at a point $\mathbf{x}$ in V can be expressed as:

$$p(\mathbf{x},\omega) = \frac{-i\omega\rho}{2\pi}\int_S v_z(\mathbf{y},\omega)\frac{\exp{(ikr)}}{r}dS(\mathbf{y}) \quad , \tag{1}$$

where ρ is the density of the fluid, ω the angular frequency, v_z the velocity in the z-direction and $r = |\mathbf{x} - \mathbf{y}|$ is the distance between the evaluation point $\mathbf{x}$ and the source $\mathbf{y}$ on the surface of S.

Equation (1) is an explicit integral expression for the radiated pressure called the Rayleigh-Sommerfeld integral. The velocity v_z is assumed to be known, and this relationship is valid for any velocity distribution over the surface. For the piston source case where the velocity $v_z = v_0(\omega)$ on S and $v_z = 0$ elsewhere, leads to:

$$p(\mathbf{x},\omega) = \frac{-i\omega\rho v_0}{2\pi}\int_S \frac{\exp{(ikr)}}{r}dS \tag{2}$$

Physically equation (2) says that the pressure wave field generated by the transducer can be considered to arise from a superposition of elementary point sources (spherical waves) distributed over the surface of the transducer.

1.3 On axis pressure

Along the central axis of a circular transducer of radius a, and using the symmetry of the problem, the integral of the equation (2) can be evaluated to obtain an analytic expression for the pressure wave field(1):

$$p(z, \omega) = \rho c v_0 [\exp{(ikz)} - \exp{(ik\sqrt{z^2 + a^2})}] \quad (3)$$

Equation (3) shows that the on axis pressure contains two propagation terms: the first one is a wave that traveled from the face of the transducer in the z direction to the evaluation point, and the second wave that has gone through a distance $(z^2 + a^2)^{1/2}$, the distance from the transducer edge to the evaluation point. These two waves are called the direct wave and edge wave, respectively.

1.4 Off axis pressure

In general, it is not possible to obtain exact analytic results for the pressure in the frequency domain for points not located on the axis of the transducer even when the transducer is circular. However, it is possible to find asymptotic expressions valid in the far field.

When $R >> \rho$, is valid to approximate $r \cong R - \rho \;\text{ sen}\, \theta \cos \phi$ (the meaning of the parameters is explained in figure (2)). Using this expression in the Rayleigh-Sommerfeld integral equation (2) and integrating the resultant Bessel function of order zero, yields an expression for the pressure in the frequency domain in terms of the evaluating point[1]:

$$p(\mathbf{x}, \omega) = -i\omega \rho v_0 a^2 \frac{\exp{(ikR)}}{R} \frac{J_1(ka \text{ sen } \theta)}{ka \text{ sen } \theta} \quad (4)$$

Equation (4) shows that the far-field off-axis behavior involves three factors. The first one is a frequency dependent coefficient; the second represents a spherical spreading wave and the third describes the angular dependence of the amplitude of this spherical wave.

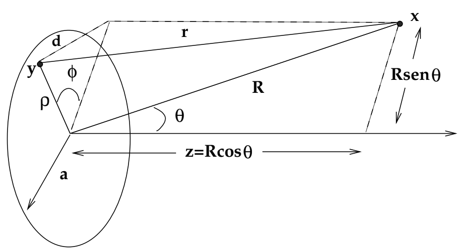

Figure 2: Geometry to calculate the off-axis far field response of a circular transducer.

1.5 Immersion Phased Array

One way to model the pressure field generated by a linear phased array transducer immersed in a fluid is to solve the Rayleigh-Sommerfeld integral equation (2) for each active element of the device and then add all the contributions using the superpositon principle. The key issue is that every contribution related to an active element of the phased array must contain a phase term that takes into account the relative delay between different elements.

In this particular case, the integrals to solve are over planar rectangles with height much larger than their width, and unlike the planar circular transducer, there is no known analytical solution.

The pressure at an arbitrary point $\mathbf{x}$ in a fluid due to a linear phased array of frequency ω with N_e active elements can be written as:

$$p(\mathbf{x},\omega) = \frac{-i\omega\rho v_0}{2\pi} \sum_{t=1}^{Ne} \int_{S_t} \frac{\exp\left(ikr_t - \phi_t\right)}{r_t} dS_t \quad , \tag{5}$$

where S_t denotes the surface of the t-th active element, r_t the distance from the evaluating point $\mathbf{x}$ to the source point in the t-th active element and ϕ_t is the phase term responsible for the delay of the t-th active element.

2 Fluid-Solid Interface

In NDE immersion testing the beam of sound generated by the probe must cross a fluid-solid boundary before it strikes a scatterer. In order to take into account the interface, the acoustic field will be expanded in plane waves.

2.1 Angular Spectrum of Plane Waves

The spherical waves present in the Rayleigh-Sommerfeld equation can be represented in terms of an angular spectrum of plane waves. Thus, using the Weyl´s integral(2) in the Rayleigh-Sommerfeld integral gives:

$$p(\mathbf{x},\omega) = \frac{\omega\rho v_0}{4\pi^2} \int_S \left\{ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\exp\left[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})\right] dp_x dp_y}{p_z} \right\} dS(\mathbf{y}) \quad , \tag{6}$$

where $\mathbf{p}$ are the wave number vectors.

Equation (6) expresses the pressure in the fluid in terms of a superposition of plane waves. These plane waves can be transmitted through a plane interface simply by using the transmission coefficients and phase terms.

2.2 Planar Interface

When a planar transducer in an immersion-testing setup radiates through the interface with a solid, mode-converted waves are present(3). The transducer generates longitudinal or compressional waves (L-waves) in the fluid, but in the solid there will be longitudinal and transverse or shear waves (T-waves). Then, using incident waves in the fluid given by equation (6) and the plane wave transmission coefficients for a fluid-solid interface, the displacement field in the solid is given by(1):

$$\mathbf{u}(\mathbf{x},\omega) = \sum_{\alpha=L,T} \frac{\omega\rho_1 v_0}{4\pi^2 i\omega\rho_2 c_{\alpha 2}} \int_S \left\{ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{T_{12}^{\alpha,L}\mathbf{d}^\alpha \exp\left[i\mathbf{p}_2^\alpha\cdot(\mathbf{x}-\mathbf{y}) + p_{1z}D - p_{2z}^\alpha D\right] dp_x dp_y}{p_{1z}} \right\} dS(\mathbf{y}) \quad , \tag{7}$$

where c_{L1} is the wave speed of the fluid; $\rho_m\,(m=1,2)$ are the densities of the fluid and solid, respectively. $T_{12}^{\alpha;L}$ are transmission coefficients (based on stress/pressure ratios) for L-waves and T-waves; $c_{\alpha 2}$, $\mathbf{d}^{\alpha}$ $(\alpha = L, T)$ are wave speeds and polarization vectors in the elastic solid for L and T-waves. The wave number vectors $\mathbf{p}_2^{\alpha}\,(\alpha = L, T)$ for L and T waves are given by:

$$\mathbf{p}_2^{\alpha} = p_x\mathbf{e}_x + p_y\mathbf{e}_y + p_{2z}^{\alpha}\mathbf{e}_z \quad ,$$

$$p_{2z} = \begin{cases} i\sqrt{p_x^2 + p_y^2 - k_{\alpha 2}^2} & \text{if} \quad p_x^2 + p_y^2 > k_{\alpha 2}^2 \\ \sqrt{k_{\alpha 2}^2 - p_x^2 - p_y^2} & \text{if} \quad p_x^2 + p_y^2 < k_{\alpha 2}^2 \end{cases} \quad , \tag{8}$$

where $k_{\alpha 2}\,(\alpha = L, T)$ is the wave number in the solid for L and T waves. Applying the stationary phase method to equation (7) gives:(1)

$$\mathbf{u}(\mathbf{x},\omega) = \sum_{\alpha=L,T} \frac{-\rho_1 v_0}{2\pi\rho_2 c_{\alpha 2}} \left\{ \int_S \frac{T_{12}^{\alpha L}(\cos\theta_1^{\alpha})\mathbf{d}^{\alpha}\exp\left[i(k_{L1}D_1^{\alpha} + k_{\alpha 2}D_2^{\alpha})\right]}{\sqrt{D_1^{\alpha} + c_{\alpha 2}D_2^{\alpha}/c_{L1}}\sqrt{D_1^{\alpha} + c_{\alpha 2}\cos^2\theta_1^{\alpha}D_2^{\alpha}/c_{L1}\cos^2\theta_2^{\alpha}}} \right\} dS(\mathbf{y}) \quad , \tag{9}$$

where $D_m^{\alpha}\,(m = 1,2)\,(\alpha = L,T)$ are distances traveled through the interface along stationary phase paths that satisfy Snell´s law for L and S waves, respectively, and $\theta_m^{\alpha}\,(m=1,2)\,(\alpha = L, T)$ are the associated angles.

Equation (9) is a high frequency approximation for the transmitted waves based on the stationary phase method. Near the interface other waves (such as surface waves and head waves) are possible in addition to waves transmitted directly through the interface along a stationary phase ray path. However the transmitted waves retained in equation (9) are likely to make the most important contributions to the wave field.

3 Numerical Models

Evaluating the fields generated by a transducer is one of the more complex tasks encountred in modeling ultrasonic systems. Therefore it is useful to consider an approach that can effectively handle complex cases. In this section the edge element method is presented.

3.1 Edge Elements

The previously obtained expressions for the beam model are surface integrals of the form:

$$I = \int_S f\exp(i\phi)dS \tag{10}$$

In the edge element approach, the surface S is divided into small planar elements $\Delta S_m\,(m = 1,2,...,M)$ and over each of these f is considered constant, f_{0m}. Furthermore the phase term ϕ is expanded in each element to first order in the form $\phi \cong \phi_{0m} + i\mathbf{p}_{0m}\cdot\mathbf{x}$ where ϕ_{0m} y $\mathbf{p}_{0m}$ are constants over each element. Using the Stokes theorem, the surface integral can be reduced to a line integral around the edges of the element. If the edges of each element are N straight line segments, the line integrals can be reduced to analytical form. Then equation (10) can be written as:

$$I = \sum_{m=1}^{M} \sum_{n=1}^{N} f_{0m} I_{mn} \exp\left(i\phi_{0m}\right) \quad , \tag{11}$$

where I_{mn} are known functions of the parameters and N is the number of straight line segments of the borders of each element. Equation (11) expresses the original surface integral in terms of the contributions from the edges of the planar elements that compose the surface S.

3.2 Planar Transducer

Using the edge element method, equation (2) can be expressed as:

$$p(\mathbf{x}, \omega) = \frac{\rho c v_0}{2\pi} \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{\exp\left(ikr_0^m\right)}{r_0^m} I_{mn} \quad , \tag{12}$$

where I_{mn} are known functions and r_0^m is the distance between the evaluation point $\mathbf{x}$ and the center of the m-th element.

This method was applied to a 1cm diameter planar transducer radiating in water at 5 MHz. The transducer surface was divided into 2700 quadrilateral elements. In figure (3) the comparison between the numerical and analytical solution [see eq.(3)] for the pressure on axis can be seen. In figure (4) the numerical and the asymptotic analytical solution for the angular diffraction pattern in the far field are plotted.

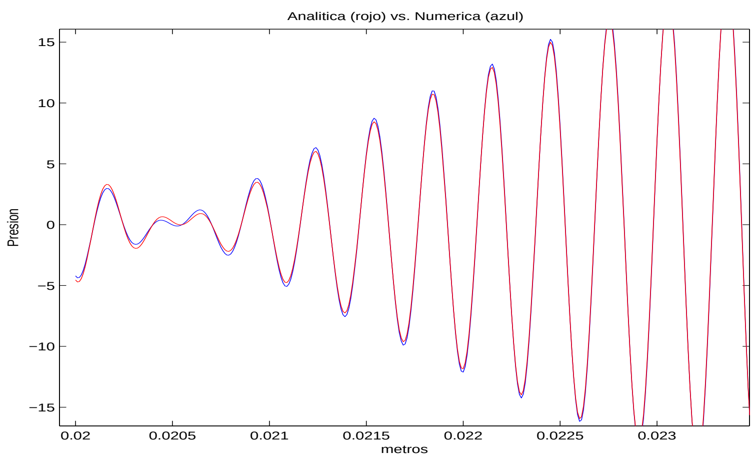

Figure 3: On axis pressure profile for a 10mm diameter planar transducer at 5 MHz. Numerical (blue) and analytical (red) solutions.

3.3 Phased Array in a Fluid

Following the same steps previously illustrated and considering the superposition principle, the pressure of a phased array device of T active elements can be written as:

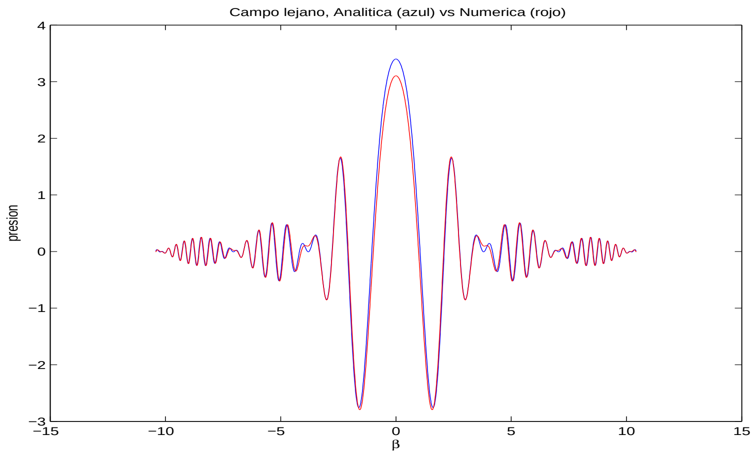

Figure 4: **Angular diffraction pattern in the far field. Numerical (blue) and asymptotic analytical (red) solutions.**

$$p(\mathbf{x},\omega) = \frac{\rho c v_0}{2\pi} \sum_{t=1}^{T} \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{\exp(ikr_{0_t}^m - \phi_t)}{r_{0_t}^m} I_{mn} \quad , \tag{13}$$

where all the parameters are defined as above. The index t denotes the t-th transducer of the device. $r_{0_t}^m$ represents the distance between the evaluation point and the center of the m-th element (of the mesh) which is located in the t-th transducer. ϕ_t is the delay of the t-th transducer. M must be understood as the number of elements of the discretization mesh over the surface of each transducer of the phased array. As before, the factor I_{mn} is known.

Using the edge element method the pressure field of an immersion linear phased array of 32 active elements radiating at 5 MHz was simulated. The surface of each transducer was meshed with 298 rectangular elements.

In figure (5) the pressure field generated by the immersion phased array can be seen. In this case the delay between active elements was fixed to cero (null delay law), thus the pressure field is very similar to the pressure field generated by only one transducer. The colormap plots the pressure as a function of position in the plane perpendicular to the phased array that splits in two the active elements. On the other hand, figure (6) shows the pressure field of the linear phased array but in this case the delays follow a linear variation.

3.4 Setting the Focal Law

The main feature of phased array ultrasonic technology is the computer-controlled excitation (amplitude and delay) of individual elements in a multielement probe. The excitation of these elements at slightly different times according to a focal law[4] can generate an ultrasonic focused beam with the possibility of modifying the beam parameters such as angle, focal distance and focal spot size.

In order to set the most suitable focal law for different applications, a possible idea is to maximize the acoustic energy in a specific spot area. Therefore, the focal law was

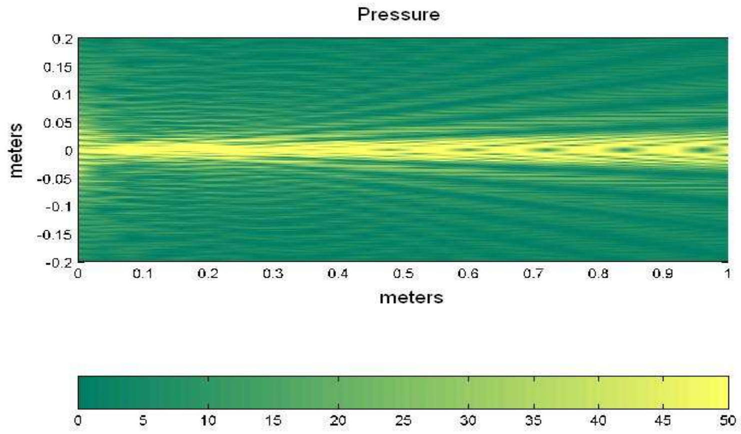

Figure 5: **Simulated pressure field of an immersion phased array of 32 active elements with a null focal law.**

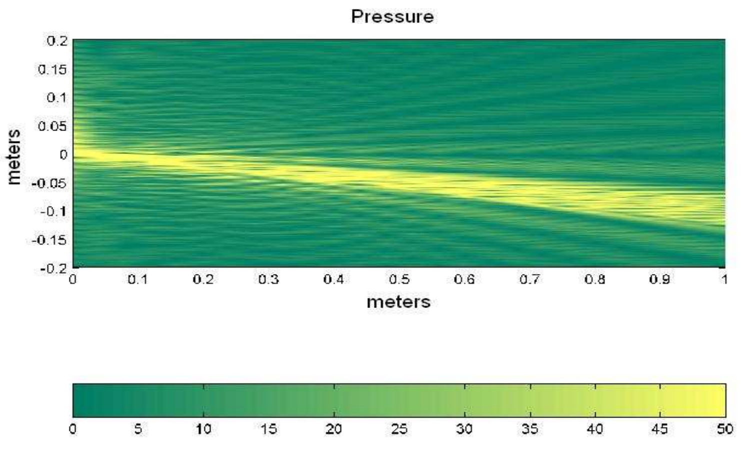

Figure 6: **Simulated pressure field of an immersion phased array of 32 active elements with a linear focal law.**

assumed to be a second order polynomial function $\phi = ax^2 + bx$ and the integral of the squared pressure over the spot area S asociated with the focal law was calculated.

In order to maximize the acoustic energy over an arbitrary region, the following function F was defined with input arguments a and b:

$$F(a, b, S) = \int_S p^2 dS \quad , \tag{14}$$

where a and b are the parameters of the focal law and S represents the spot area where the beam must be focused. Therefore is possible to use an algorithm to find parameters a and b that maximize the objective function F. In this work the simplex method for several variables by Nelder and Mead[(5)] was used to determine the optimal coefficients. The main drawback of this method is the possibility of converging to a local minimum that is far from the optimal value

In figure (7) the pressure field generated by an immersion linear phased array of 128 active elements is shown. The acoustic energy was focused in a square of 1cm side with coordinates center (-6,21). On the other hand the optimization algorithm was used for the same phased array to focus the acoustic energy in a square of 1cm side located 20 cm from the device in normal direction. The achieved pressure field can be seen in figure (8).

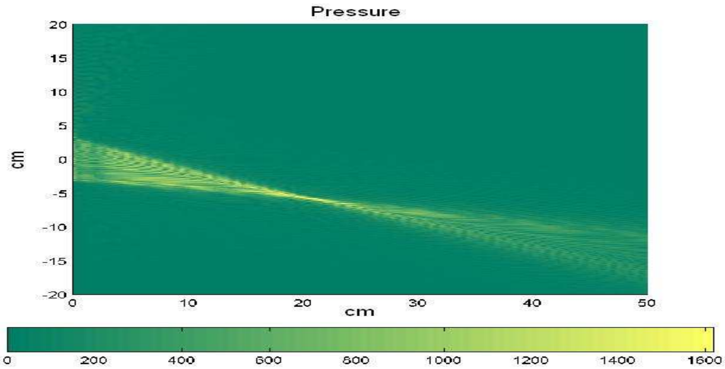

Figure 7: **Simulation of the pressure field of an immersion phased array of 128 active elements. The focal law was set with an optimization solver in order to maximize the acoustic energy in a square of 1cm side.**

3.5 Fluid-Solid Planar Interface

Applying the edge element approach to discretize the equation (7), the displacement field in the solid for a planar transducer radiating to a plane fluid-solid interface can be written as:[(1)]

$$\mathbf{u}(\mathbf{x}, \omega) = \sum_{\alpha=L,T} \frac{-i\rho_1 v_0}{2\pi k_{L1}\rho_2 c_{\alpha 2}} \sum_{m=1}^{M}\sum_{n=1}^{M} \left\{ \frac{T_{12}^{\alpha L}(\Theta_{10}^{\alpha m})\mathbf{d}_0^{\alpha m} \exp\left[i(k_{L1}D_{10}^{\alpha m} + k_{\alpha 2}D_{20}^{\alpha m})\right]}{\sqrt{\Delta_{x0}^{\alpha m}}\sqrt{\Delta_{y0}^{\alpha m}}} I_{mn}^{\alpha} \right\} \quad , \tag{15}$$

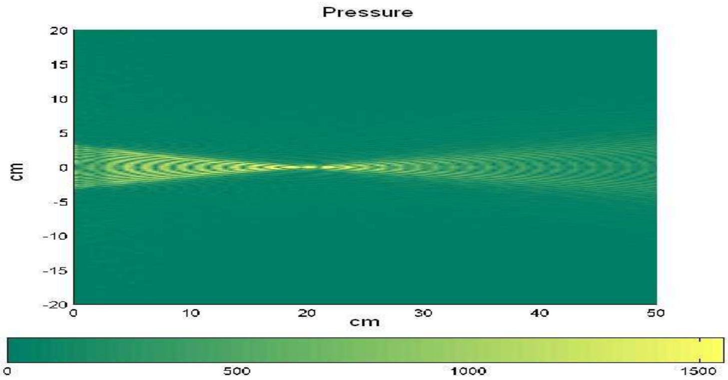

Figure 8: **Simulation of the pressure field of an immersion phased array of 128 active elements. The focal law was set with an optimization solver in order to maximize the acoustic energy in a square of 1cm side.**

where

$$\Delta_{x0}^{\alpha m} = D_{10}^{\alpha m} + \frac{c_{\alpha 2}\cos^2\Theta_{10}^{\alpha m}}{c_{L1}\cos^2\Theta_{20}^{\alpha m}} D_{20}^{\alpha m} \quad ,$$

$$\Delta_{y0}^{\alpha m} = D_{10}^{\alpha m} + \frac{c_{\alpha 2}}{c_{L1}} D_{20}^{\alpha m} \quad , \tag{16}$$

and I_{mn}^{α} are known functions of the parameters.

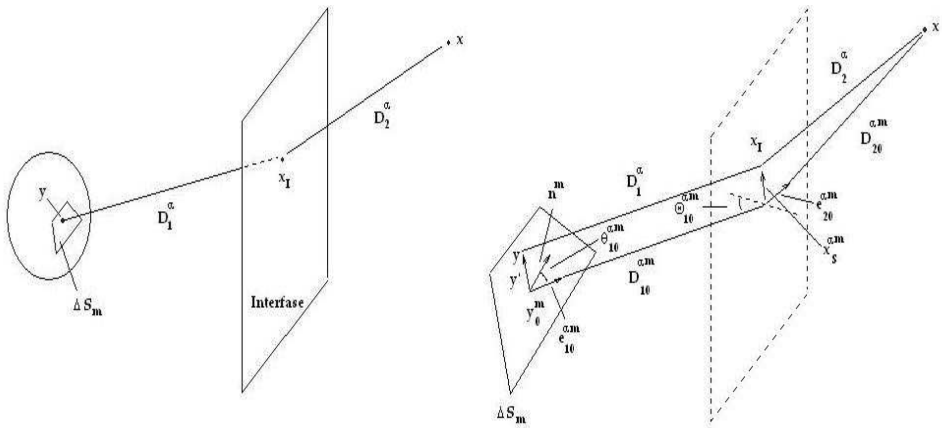

Figure 9: **Ray paths through a fluid-solid interface from a small planar element.**

3.6 Immersion Phased Array and Planar Interface

The previously used ideas were extended to calculate the displacement field in a solid due to the action of an immersion phased array.

In order to set the most suitable focal law, the acoustic energy in a specific spot area was maximized. As before the focal law was considered as a second order polynomial function of the form $\phi = ax^2 + bx$.

In this case the objective function F to maximize with input arguments a and b was defined as:

$$F(a, b, S) = \int_S |\mathbf{u}| \, dS \quad , \tag{17}$$

where a and b are the parameters of the focal law; S represents the spot area where the beam must be focused. As before, an optimization algorithm[5] was used to determine the optimal coefficients a and b.

In figure (10) the total displacement field generated by an immersion linear phased array of 32 active elements radiating at 5 MHz is shown. The acoustic energy was focused in a square of 1cm side located at 2cm from the interface and 4cm below the phased array position.

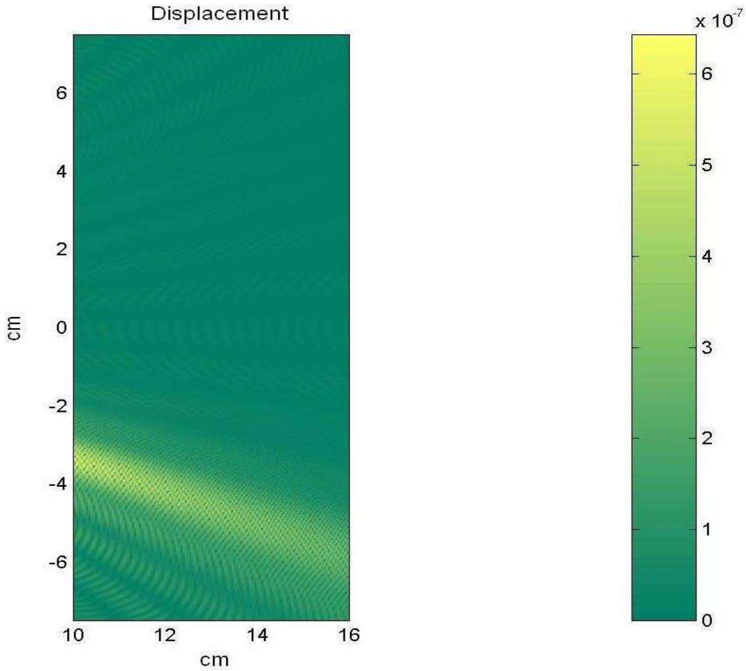

Figure 10: **Modulus of the total displacement in the solid generated by an immersion phased array of 32 active elements. The focal law was set through an optimization algorithm that maximizes the integral of the absolute value of the displacements over a fixed area.**

4 Conclusions

In a non destructive evaluation system of ultrasonic inspection the generation and propagation of waves, the interaction of these waves with some flaw in the structure of the evaluated component, the reception and interpretation of the signal from a flaw are phenomena that involve high complexity physical processes. For this reason, the modeling of the entire ultrasonic measurement process has turned into a useful tool to understand the complexity of the physical processes, to design applications, and to interpret the information in a quantitative way.

The ideas used in this work, as the Rayleigh-Sommerfeld integral and the edge elements approach to solve integral expressions, have demonstrated to be an efficient tool to model the pressure field in a fluid generated by monolithic or phased array transducers of diverse geometry.

The integral expressions can be approximated by different numerical approaches. The edge elements method is a precise and efficient tool for the calculation of pressure fields in fluids and displacements in a solids.

The proposed method to determine focal laws of phased array transducers from an iterative optimization algorithm is feasible and shows satisfactory results in the considered cases. Nevertheless, the initial value of the quadratic polynomial coefficients that describe the focal law need to be close to the optimal solution value. Otherwise it is possible that the solver finishes in a relative extreme value without relevancy.

In order to develop a complete model able to simulate the whole ultrasonic inspection process, it is suitable to decompose the measurement process into three different parts that could be approached with different methods.

The first one, which is considered in part in the present work, must describe the pressure field in the evaluating component generated by the inspection device. The components analyzed in this work were, in all cases, semi-infinite. To enrich the described model, it is necessary to develop a description of the pressure field inside bounded components, taking into account the multiple reflections of the acoustic waves inside the piece.

The second part of the model should be the description of the interaction between the incident field and the flaw.

Finally, the third and last part of the model would have to consider the reception and interpretation of the signal measured by the instrument.

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