P-τ TRANSFORMATION AS THE EFFICIENT TOOL FOR DETERMINATION OF THE VELOCITY DISPERSION CHARACTERISTICS IN COMPLEX STRUCTURES

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Abstract: Last years show the increasing interest in using Spectral Analysis of Surface Waves (SASW) for the NDE of concrete structures. The dispersion characteristics are there commonly determined using the classic approach based on the Phase Difference measurements (PD) method. Meanwhile, this approach is accurate only for simplest cases. When the investigated signal include different components (modes) if different velocities and if these modes overlap in time domain, the use of the PD may lead to serious errors.

In preliminary studies the authors emphasized on the interest of using the surface waves for determination of the near surface deterioration of concrete. This deterioration may be equivalent to the two-layered structure which involves the modal propagation. In order to determine the dispersion characteristic of each mode, the so-called (P-τ) transformation is proposed. The developed processing code is validated using the simulated input signals with known dispersion law. Then the accuracy of the (P-τ) is studied as the function of the acquisition system parameters like receiver spacing, receiver number, aperture length, sampling rate and signal frequency. The easy to use criteria are formulated.

Introduction: The well known free-space Green function for the elementary case illustrated in Fig.1.a (Kinsler et Frey 1982) yields:

\[
G(M, R, \omega, t) = \frac{e^{-jkR}}{4\pi R} e^{i\omega t} \tag{1.a}
\]

The function \(G\) represents the signal observed at the distance \(R\) and radiated in unlossy free-space from a point source \(S\), placed in \(M\) and excited by harmonic signal \(e^{i\omega t}\); \(k = \omega / V\) is the wave number and \(V\) denotes propagation velocity. For pulsed excitation \(\delta(t)\) above solution becomes (Harris,1981):

\[
g(M, R, t) = \mathcal{F}^{-1} \{ G(M, R, \omega) \} = \frac{\delta(t - R / V)}{4\pi R} \tag{1.b}
\]

where symbol \(\mathcal{F}\) indicates the Fourier transformation. Consequently, signal \(x(R, t)\) obtained for the excitation \(s(t)\) can be found by means of the convolution:

\[
x(R, t) = s(t) * g(R, t) = \frac{1}{4\pi R} x(t - R / V) \tag{2}
\]

As illustrated in Fig.1.b, velocity \(V\) can be determined in time domain, measuring delay \(\Delta t\) of signals \(x(R_1, t)\) and \(x(R_2, t)\). Such measurement yields the group velocity \(V_g\), velocity which characterizes the total signal. From Eq.(2) the delay can be found as \(\Delta t_g = \frac{R_1}{V_g} - \frac{R_2}{V_g} = \frac{\Delta R}{V_g}\) and finally the group velocity can be found as:
In practice, for pulsed signals, the delay $\Delta t$ is found either taking first arrivals instants of signals ($\Delta t_e$; easy to detect), or taking the delay $\Delta t_e$ between their most significant amplitudes (Fig.1.b). In the latter case $V_g$ is in fact supposed to be equal to so called energy velocity $V_e$ which is the velocity of the most energetic signal component.

The signal $x$ can be analyzed in the frequency domain as

$$X(j\omega) = \mathcal{F}\{x(t,R)\} = |X(\omega, R)| e^{-j\phi(\omega)}$$

and

$$S(j\omega) = \mathcal{F}\{s(t)\}$$

Using the Fourier transform property $\mathcal{F}(s-\tau) \Leftrightarrow S(j\omega) e^{j\omega\tau}$ the phase $\phi$ term may be found as:

$$\phi(\omega, R) = \frac{\omega R}{V} = kR$$

Finally, velocity $V$ can be found using the setup shown in Fig.1.b, by measuring the phase difference $\Delta\phi = \phi_2 - \phi_1$ of signals $x(R_1, t)$ and $x(R_2, t)$:

$$V_{ph}(\omega) = \frac{\omega (R_2 - R_1)}{\Delta\phi(\omega)} = \frac{\omega \Delta R}{\Delta\phi(\omega)}$$

The velocity found using this way known as phase velocity $V_{ph}(\omega)$ and the method based on relations (5) and (6) will be further referred as phase difference method (PD). For the no dispersive medium the group and phase velocity are equal i.e $V_g = V_{ph}$. In the contrary, when the dispersion of the medium the group and phase velocity are equal i.e $V_g = V_{ph}$. In the contrary, when velocity dispersion occurs, the phase velocity is not constant and becomes a function of frequency. Then Eq. (5) assumes more general form:

$$\phi(\omega, R) = \frac{\omega R}{V_{ph}(\omega)},$$

and function $V_{ph}(\omega)$ represents the velocity dispersion characteristic. Notice, that if Eq.(6) is used in order to determine $V_{ph}(\omega)$, then, the phase periodicity yields the infinity of solutions:

$$\Delta\phi(\omega, R) = n2\pi + \Delta\phi_0(\omega)$$

In order to find the unique solution, modulo $2\pi$ term should be removed. Then $V_{ph}$ is found as:

$$V_{ph}(\omega) = \frac{\omega (R_2 - R_1)}{\Delta\phi_0(\omega)}$$
The PD method is basic and quite popular in NDE field (see for example Cho, 2003; Mathews, 1996; Hassaim et al. 2001; Goueygou, 2003)). Its procedure is illustrated in Fig. 2: the input pulses \( x_1 \) and \( x_2 \), were modeled numerically, for the dispersive medium characterized by the velocity dispersion \( V_{ph}(\omega) \) taken as:

\[
\frac{1}{V_{ph}(\omega)} = \frac{1}{V_0(\omega_0)} - \frac{\alpha_0}{\pi^2} \ln \left( \frac{\omega}{\omega_0} \right)
\]

where \( V_0 \) indicates the highest (i.e. group) velocity [above relation characterize the medium with the absorption increasing proportionally to the frequency \( \alpha=\alpha_f \) (Azini, 1968)]. Fig.2.a shows two signals \( x_1 \) and \( x_2 \) computed for \( \alpha_o=25\text{dB/mMHz} \), \( V_o=1000 \text{ m/s} \) and \( s(t)=\delta(t) \). The computations were performed using the DR method (Piwakowski and Sbai, 1999 ; Piwakowski and Lingvall, 2004). Fig.2.b shows phases \( \phi_1 \) et \( \phi_2 \) which display, as it could be expected from Eq.(8), the \( 2\pi \) periodicity. Section (c) shows the \( V_{ph}(\omega) \) obtained which agrees perfectly with the relation (10) confirming the accuracy of the method.

In spite of advantages its advantages, however, the PD approach also displays several disadvantages:

1. gives uncertain results when signal to noise ratio is poor,
2. the removing of the phase periodicity may lead to certain errors,
3. PD method is not adapted to the more complicated situations occurring when input signals include the different components (modes) which propagate with different velocities. For example this occurs when the structure is layered (modes of Lowe and pseudo-Rayleigh waves). If modes overlap in time their separation in time domain is impossible and use of the PD method may lead to serious errors.

The problem mentioned in (3) is illustrated in Fig 3 on example of the real data. Fig.3.a shows the five input pulses \( x_1...x_5 \) recorded at distances \( R_1...R_5 \), windowed which Gaussian window in order remove the noise and the not desired signal parts. The input pulses include the different modes displaying the different velocities which overlap in time (i.e. not seen directly). Finally each pair of signals yields different dispersion characteristics (section (b)) and the averaging of the obtained characteristics obviously does not improves the accuracy and leads to the erroneous result without any physical meaning.
P-τ transformation: In order to overcome the above mentioned errors the P-τ transformation, known also as the Slant Stack (SL) transformation, used in geophysics, especially in MASW techniques (multichannel analysis of surface waves). (Ymaz, 1987, Leparoux, 2002). The input N signals recorded in positions R₁ ...Rₙ are processed as the two dimensional signal \( x(t,R) \). The SL procedure consists in introduction of the linear delay \( \tau = R/V = pR \):

\[
y(\tau, R, p) = x(t - pR, R)
\]

where the parameter \( p \) indicates the slowness:

\[
p = \frac{1}{V}
\]

The operation is performed in the interval of interest \( p_{\text{min}} - p_{\text{max}} \). Summing all \( y \) in \( R \) domain yields the P-τ gather which represents the input data in the slowness-arrival time domain:

\[
P_τ(\tau, p) = \sum_{R₁}^{Rₙ} x(\tau, R, p)
\]

The time domain Fourier transformation of \( P_τ \) and relation (12) convert the input data into \( (V,\omega) \) domain:

\[
SL(\omega, V) = \Im \{ P_τ(\tau, p) \} V = 1/p
\]

Finally the researched dispersion characteristic \( V_{ph}(\omega) \) is obtained as

\[
V_{ph}(\omega) = \max_{\omega} \{ ST(\omega, V) \}
\]

Figure 4 shows an example of the use of the SL transformation. The six input pulses \( x₁...x₆ \) computed by the same mean as those used in Fig.2, are shown in Fig.4.a. \( P_τ \) plot detects the arrival instant of the input data set (\( \tau_a = 30 \) ms) and shows the interval of detected velocities. The SL plot shows the velocity interval and the frequency bandwidth of input data. The dispersion characteristic found using Eq.(14) perfectly agrees with that defined by Eq.(10).
Figure 5 illustrates the SL transformation applied to the real measured signal which contains two modes differing in velocity and frequency. Two modes are clearly separated and their separate dispersion characteristics are obtained. Comparing with the FD method the SL transformation offers the following advantages:

- $V_{ph}(\omega)$ is obtained from total set of data, and not from two signals only, thus the signal/noise ratio is improved and the averaging of individual characteristics is not necessary.
- different modes can be visualized, identified, and analyzed both in $(V, \omega)$ and in $(V, t)$ domains,
- the operations (17) and (18) are reversible thus allowing a mode extraction and return into time domain
- the group and energy velocity can be determined in the same time, directly from the SL plot.
Accuracy: The problem of accuracy of SL transformation is generally ignored by its users. In the same time several authors consider the approach SL as to be a aliasing free, and this statement seems be physically doubtful.

The included examples show that the accuracy of the determination of the measured velocity is always limited and is directly related with the width of cross section of SL in velocity domain. In this section the accuracy of the SL method will be formally evaluated. In order to study this we introduce the velocity amplitude function $A(V, V_s)$ defined as the cross section of SL plot obtained for the input signal having constant velocity $V_s$ and constant frequency $\omega=\omega_s$.

$$A(V, V_s) = SL(V, \omega = \omega_s, V_{\text{input}} = V_s)$$

(15)

The SL procedure can modeled as the $N$ element linear array of length $L$, receiving the planewave $x(t, k) = e^{j(\omega t - k \cdot x)} : k = \omega_s / V_s$, of wavelength $\lambda_s = V_s / f_s$, arriving from direction $\theta = \pi / 2$, (Fig.6).

Modeling the operations expressed in Eqs.(11),(12) and (13) using the linear array model the array output $A(V, V_s)$ yields:

$$A(V, V_s) = \frac{\sin(N \omega dB)}{N \sin(\frac{\omega dB}{2})}$$

(16)

where $B = (1 / V_s - 1 / V)$ and $d$ is the element spacing.

As it can expected from the array theory the $A(V)$ is the function of two main array parameters: the spatial sampling density $d/\lambda_s$ and the array length with regard to the wavelength $L/\lambda_s$. The figures 7 and 8 show the $A(V)$ plots as the function of the above two parameters. The analysis of these results leads to the following conclusions:

- if the spatial sampling condition criterion $d/\lambda_s < 0.5$ is not fulfilled the SL transformation generates the accurate result which is accompanied by false results (ghost images) and their number increases when $d/\lambda_s$ increases (Fig.7.a). If undersampling is important the accurate result may even interfere with the false results (Fig.7.d).
- Taking into account the evident relation $\lambda = V / f_s$, and that the analysis is performed as a function of $V$, the sampling criterion varies during analysis. Finally the results can be corrupted for $d < V / 2 f_s$ (i.e. under the line seen in Fig.7.b,c,d) or in velocity interval defined as $V < d / 2 f_s$ (see lines shown in Fig.7.e and Fig.8.e and the ghost images present under these lines).
- Figure 8 provides the study of $\Delta V_{3dB}$ as the function of array length/wavelength ratio. As it could be expected, accuracy increases when $Nd/\lambda_s$ increases. By analogy with the filter theory, we define the 3dB width $\Delta V_{3dB}$ of the main lobe of $A(V)$ observed in vicinity of $V_s$ as the accuracy measure ($\Delta V_{3dB}$ is illustrated in Fig.8.b). The solution (16) enables to approximate accuracy as:
The above formula provides the precision close to 2% when $\Delta V_{3dB}/V_s$ is smaller than 5%. The plot of the ratio $\Delta V_{3dB}/V_s$ is shown in Fig. 8.d. It may be concluded that the accuracy better than 10% requires the array length $L/\lambda$ better than to 10.

**Conclusions:** The performances of the SL transform has been evaluated and accuracy of the velocity estimation was studied and expressed quantitatively, as a function of the array sampling density and the array length in regard to wavelength.

It is shown the SL transformation is not aliasing free, as it is reported in certain references. In fact even if the sampling criterion is not fulfilled, the result can be correct, but it will be accompanied by the false detections of the false, not existing velocities. Thus effect is difficult to avoid because the sampling criterion varies with velocity which, by principle, varies during the analysis. Finally during the use of the SL method, the sampling criterion should controlled by the operator, for example by drawing the sampling limit curve $V=2df$ like it was done in Figs. 7 et 8.

The accuracy better then 10% requires the array length/wavelength ratio to be of order of ten.
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