Fast and Analytical Exact Reconstruction of Large CT-Volumes

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Abstract
Computed Tomography (CT) has become very important for industrial applications. The fields of application are ranging from highly specialized tasks in CT inline inspection to universal X-ray systems, metrology and high-resolution CT, for example in microelectronics. For high-energy scans of high quality fan-beam CT using linear detector arrays (LDA) is the right choice due to the bigger field of view and the superior collimation reducing scatter artefacts compared to flat panel detectors, which are used for cone-beam CT.

The conventional filtered back projection (FBP) as reconstruction algorithm requires equally spaced X-rays, i.e. it was designed for parallel beam geometry. Its computational effort depends mainly on the size of the reconstructed volume.

In this article we will introduce a fast and analytical exact reconstruction algorithm, based on the orthogonal polynomial expansion on the disk (OPED). Its numerical complexity only depends on the amount of the input data (projections). OPED is designed to overcome aliasing artefacts and streaks. It needs no special filtering, i.e. it can be easily parameterized. The alignment of the sample grid is very well suited for fan beam geometry with its non-equally spaced X-rays. As result, we will compare the performance of the OPED and FBP algorithm.

Keywords: Computed Tomography (CT), Line Detector Array (LDA), automotive, aerospace, Large CT-Volumes, Orthogonal Polynomial Expansion on the Disk (OPED), Filtered Back Projection (FBP)

1. Introduction

Computed Tomography (CT) has become very important for industrial applications. The main advantage of CT is the visualization of material density information in the three-dimensional space. In general this information can be mapped one-to-one to the test part directly, which has also a three-dimensional extent. In comparison, radiographic images consist of an overlay of all density information along the beam direction. Therefore the real position of density variacnces in space can be only depicted with a CT scan correctly.

The fields of application are ranging from highly specialized tasks in CT inline inspection to universal X-ray systems, metrology and high-resolution CT, for example in microelectronics. In any field the acquired CT data will help either to speed up the inspection process e.g. for initial prototyping or to reduce the number of rejects for safety-critical parts, where simple radiography will lead to ambiguous results.

The reconstruction process itself consists of the acquisition of several digital X-ray images from different viewing angles, so called projections, and the mathematical computation of certain slices. The state-of-the-art reconstruction algorithm, the filtered backprojection (FBP, [5]), needs projections, which are acquired from equally spaced segments of a circle perpendicular to the rotation axis of the test part. Moreover the FBP as reconstruction algorithm requires equally spaced X-rays, i.e. it was designed for parallel-beam geometry originally. Its computational effort depends mainly on the size of the reconstructed volume.
Today the computational effort, i.e. the reconstruction time, becomes less important due to the hardware acceleration because of multi-kernel central processing units (CPU) or affordable graphic boards.

Without loss of generality, in this article we will concentrate on fan-beam geometry, i.e. the reconstruction result, called tomogram, is always one slice perpendicular to the rotation axis. Anyway, the presented algorithms themselves are expandable to the three-dimensional space and are used for cone-beam CT to reconstruct several slices or even the whole volume in one reconstruction process.

Particularly for high-energy scans of high quality fan-beam CT linear detector arrays (LDA) are used due to the bigger field of view and the superior collimation reducing scatter artefacts compared to flat panel detectors, which are used for cone-beam CT applications.

In the next section we will introduce first a fast and analytical exact reconstruction algorithm, based on the orthogonal polynomial expansion on the disk (OPED). Its numerical complexity only depends on the amount of the input data (projections). OPED is designed to overcome aliasing artefacts and streaks. It needs no special filtering like FBP, i.e. it can be easily parameterized. The alignment of the sample grid is very well suited for fan beam geometry with its non-equally spaced X-rays.

In the result section we will compare the performance of the OPED and FBP algorithm explicitly and in the final section we will summarize and rate the comparison of the two competing algorithms.

2. CT-Reconstruction

2.1 Formula of OPED

The FBP inversion formula can be obtained with the help of a special relationship between the Fourier transforms of the object function and its Radon transform. This relation is called Fourier Slice Theorem [1]. Representing a function in the basis of orthogonal polynomials, completely different inversion formulae can be obtained. The fundamental role within this approach is played by the relation

\[
p_q(Q;t) = \frac{2}{k+1} \sqrt{1-t^2} U_k(t) \frac{Q(\cos \phi, \sin \phi)}{(k+1) \theta}
\]

where \( U_k(t) = \sin((k+1)\theta)/\sin \theta \), \( t = \cos \theta \), is Chebyshev polynomial of the second kind, \( p_q(Q;t) \) is the Radon projection of \( Q \), and \( Q \) is a bivariate orthogonal polynomial of degree \( k \) (see [2]). With the help of (1) and using interpolating properties of polynomials, the inversion formula

\[
f(x, y) = \sum_{k=0}^{\infty} (k+1) \frac{1}{\pi} \int_{0}^{\pi} c_k(\phi) U_k(x \cos \phi + y \sin \phi) d\phi
\]

where
\[
c_k(\phi) = \frac{1}{\pi} \int_{-1}^{1} p_\phi(f; t) U_k(t) dt
\]

(3)

can be obtained (see [3]). In [3] it was shown that the best polynomial of degree \( n \) that approximates the object function is given by

\[
A_n f(x, y) = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\nu=0}^{n-1} (k+1) c_k(\phi) U_k(x \cos \phi + y \sin \phi),
\]

(4)

\[
\phi_v = \frac{2\pi v}{n}, \quad 0 \leq v < n.
\]

(5)

The inversion formula (4) is called Orthogonal Polynomial Expansion on Disk (OPED) due to the formalism that has been used for its derivation. As one can see from the structure of OPED, the Radon data appear in form of projections measured at discrete angles (5). In practice, projections are not continuous either. Therefore, in order to calculate the coefficients \( c_k(\phi_v) \), the integral

\[
\int_{-1}^{1} p_\phi(f; t) U_k(t) dt = \int_{0}^{\pi} p_\psi(\psi) \sin(k+1)\psi d\psi,
\]

(6)

has to be approximated by an appropriate quadrature. The best result is expected when the quadrature relates to the regular sampling of the parameter \( \psi \) (see [3], [4]). Hence, sampling conditions that must be imposed on the Radon transform are defined by the order of a bivariate polynomial that is supposed to play the role of interpolating polynomial for the object function. The corresponding sampling lattice is the standard lattice in the plane coordinated by \((\phi, \psi)\). We note that the data collected within this scanning geometry can be reordered to fan-beam data and vice versa. Thus OPED can also be used for the reconstruction from data collected within the fan-beam geometry. Clearly, in order to do this, the fan-beam data have to be reordered appropriately.

2.2 Scanning Geometry

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{parallel_projection.png}
\caption{Parallel projection geometry. Left: OPED-Projection, right: FBP-Projection}
\end{figure}

At the left in Figure 1 there is a set of parallel rays obtained after reordering the rays collected within the fan-beam geometry. Rays within this set are characterized by fixed \( \phi \) and constant sampling rate \( \Delta \psi \) as required for the approximation of the integral in (6). For comparison at
the right there are parallel rays, which are distributed regularly at sampling rate $\Delta t$. The data collected within this geometry can be reconstructed with parallel-beam FBP. However fan-beam data cannot be reordered to parallel-beam data distributed regularly in terms of $t$. Therefore the numerically more expensive fan-beam FBP has to be used.

### 2.3 Advantages of OPED

In the beginning of this article we introduced OPED as fast and analytical exact reconstruction algorithm. The theoretical part above lead to the following two main advantages of the OPED:

1. **Data geometry.** OPED works with data collected within the parallel-beam geometry with sinusoidal lateral sampling. Under certain conditions such data can be obtained from the fan-beam data via simple reordering. In general fan-beam data can be reordered to the data consisting of $n$ semi-parallel projections. This fact has a positive effect on the speed of the reconstruction.

2. **The OPED returns the interpolating polynomial of degree that depends on the number of projections.** This polynomial is the best among all polynomials of this degree approximating the object function in the sense of squared deviation.

In the following section we will prove this theoretical motivation of the OPED by a practical comparison between a state-of-the-art FBP implementation and an implementation of the OPED algorithm.

### 3. Results

#### 3.1 Performance measurements

In the following sub-sections we will compare the performance of the newly introduced OPED and the conventional, state-of-the-art FBP algorithm. First we will measure the computational effort, but we will also measure image quality parameters to get an objective comparison.

##### 3.1.1 Computational effort

As we mentioned already, the OPED’s numerical complexity mainly depends on the amount of the input data, especially the number of projections. In comparison the complexity of the FBP depends mainly on the size of the reconstructed volume and the number of projections.

In Table 1 the dimension of the input data in pixel, i.e. the length of the detector per number of projections, the size of the tomogram in pixel and the dedicated reconstruction times in seconds are given. From the measured reconstruction times the mentioned relations between complexity and the chosen algorithm can be verified easily. All measurements were performed on an Intel Core i7 870 CPU at 2.93 GHz and 16 GB memory on Microsoft Windows 7, 64 Bit, i.e. each algorithm used eight processor kernels in parallel.

From Table 1 it becomes clear that the OPED is well suited for large numbers of projections and the dedicated large reconstruction grid. For 4164 projections the OPED is nearly three times faster than the FBP at a typical reconstruction grid size of $4096^2$ pixels. For 2082 projections the OPED is 2.5 times faster than the FBP at a typical reconstruction grid size of...
2048² pixels. Even for 1440 projections the OPED is up to three times faster than the FBP at a typical reconstruction grid size of 1024² pixels or 2048² pixels.

Table 1. Computational effort regarding size of input data and tomogram

<table>
<thead>
<tr>
<th>Input Data [pixel]</th>
<th>Tomogram [pixel]</th>
<th>OPED [s]</th>
<th>FBP [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>500 x 4164</td>
<td>512 x 512</td>
<td>70.20</td>
<td>3.94</td>
</tr>
<tr>
<td></td>
<td>1024 x 1024</td>
<td>70.39</td>
<td>13.05</td>
</tr>
<tr>
<td></td>
<td>2048 x 2048</td>
<td>70.28</td>
<td>50.41</td>
</tr>
<tr>
<td></td>
<td>4096 x 4096</td>
<td>70.80</td>
<td>200.41</td>
</tr>
<tr>
<td>1000 x 4164</td>
<td>512 x 512</td>
<td>71.97</td>
<td>4.16</td>
</tr>
<tr>
<td></td>
<td>1024 x 1024</td>
<td>72.29</td>
<td>13.25</td>
</tr>
<tr>
<td></td>
<td>2048 x 2048</td>
<td>72.67</td>
<td>50.84</td>
</tr>
<tr>
<td></td>
<td>4096 x 4096</td>
<td>73.12</td>
<td>198.66</td>
</tr>
<tr>
<td>1000 x 2082</td>
<td>512 x 512</td>
<td>10.07</td>
<td>2.28</td>
</tr>
<tr>
<td></td>
<td>1024 x 1024</td>
<td>10.12</td>
<td>6.66</td>
</tr>
<tr>
<td></td>
<td>2048 x 2048</td>
<td>10.34</td>
<td>25.25</td>
</tr>
<tr>
<td></td>
<td>4096 x 4096</td>
<td>11.19</td>
<td>39.47</td>
</tr>
<tr>
<td>1612 x 1440</td>
<td>512 x 512</td>
<td>4.88</td>
<td>1.98</td>
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<td></td>
<td>1024 x 1024</td>
<td>4.90</td>
<td>5.14</td>
</tr>
<tr>
<td></td>
<td>2048 x 2048</td>
<td>5.07</td>
<td>17.79</td>
</tr>
<tr>
<td></td>
<td>4096 x 4096</td>
<td>5.96</td>
<td>69.87</td>
</tr>
</tbody>
</table>

3.1.2 Spatial resolution

The American Society for Testing and Materials (ASTM) standard E 1695 proposes the evaluation of the Modulation Transfer Function (MTF) of a cylindrical test part as a measure of the spatial resolution of reconstructed tomograms, which will be given in Line-pairs per millimeter (Lp/mm).

The spatial resolution depends on the chosen reconstruction algorithm. In case of the FBP there exists a parameter, which controls the filtering of the input data during the reconstruction process. By default this parameter is 1.0. If this parameter becomes smaller the resulting tomogram will get a smoother look, whereas if this parameter becomes greater than 1.0 the small details or even the noise will be enhanced. This behaviour can be measured by the Signal-to-Noise-Ratio (SNR), i.e. the higher the SNR the smoother the surface. Unfortunately the smoothing of the FBP results also in a smoothing of the edges, i.e. details may be blurred. Therefore the spatial resolution behaves contrarily, i.e. the higher the SNR of the FBP tomogram the smaller the spatial resolution (see [6]).

In the following test we computed the MTF of an aluminium cylinder for FBP at different filter parameters and for OPED, which does not need a filter parameter. Afterwards we measured the spatial resolution by taking the frequency at 10% modulation of the MTF as it is proposed in the ASTM standard. The fan-beam angle regarding the cylinder diameter was approximately 9° and the detector pitch 0.254 mm.

To compare the FBP and OPED regarding the spatial resolution, we measured the SNR at the center region of the dedicated tomograms for both algorithms additionally. In Table 2 the mean spatial resolutions inclusive their standard deviations and the dedicated SNR values are shown.
Table 2. SNR and Spatial Resolution comparison of FBP and OPED

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Filter parameter</th>
<th>SNR</th>
<th>Spatial Resolution [Lp/mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>FBP</td>
<td>0.5</td>
<td>50.0</td>
<td>2.00 ± 0.02</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>22.8</td>
<td>3.06 ± 0.08</td>
</tr>
<tr>
<td></td>
<td>1.3</td>
<td>17.2</td>
<td>3.30 ± 0.15</td>
</tr>
<tr>
<td></td>
<td>1.4</td>
<td>16.2</td>
<td>3.37 ± 0.19</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>15.9</td>
<td>3.44 ± 0.22</td>
</tr>
<tr>
<td></td>
<td>1.6</td>
<td>14.7</td>
<td>3.50 ± 0.26</td>
</tr>
<tr>
<td></td>
<td>5.0</td>
<td>10.4</td>
<td>4.44 ± 0.36</td>
</tr>
<tr>
<td>OPED</td>
<td>none</td>
<td>15.9</td>
<td>3.24 ± 0.10</td>
</tr>
</tbody>
</table>

The OPED has a mean spatial resolution of 3.24 Lp/mm at a standard deviation of 0.1 Lp/mm, whereas the FBP at the default filter parameter of 1.0 has a spatial resolution of 3.06 Lp/mm and a standard deviation of 0.08 Lp/mm. In this case the SNR is higher for the FBP, but the spatial resolution is 5% better for OPED.

If the filter parameter is increased to 1.5 for the FBP, the SNR of OPED and FBP become comparable. Unfortunately the standard deviation of the spatial resolution of the FBP is also increased, i.e. the visual image quality is not visibly better than for the OPED, because of the enhanced structural noise. In this case FBP has a better spatial resolution.

### 3.1.3 Evaluation of the Point Spread Function

The measurement of the MTF in 3.1.2 is superimposed by artefacts like beam hardening or scatter. To minimize these artefacts we measured the point spread function (PSF) of a cut through a thin copper wire, whose diameter was smaller than the physical resolution of the detector (detector pitch: 0.2 mm, wire diameter: 0.18 mm).

![Figure 2. MTF for copper wire near the isocenter](image)
Then we evaluated the spatial resolution of the reconstructed tomogram by applying fast Fourier transform to a Line Spread Function (LSF) taken from the reconstructed PSF. We have done this procedure twice. First time we positioned the wire near the isocenter and second time near the border of the field of view.

Afterwards we reconstructed the input data by OPED as well as by FBP regarding filter parameter 1 and 1.5. Once the images were reconstructed the LSF was computed by integrating a small Region of Interest (ROI) near the wire as described in [7]. MTFs were subsequently evaluated as seen in Figure 2 and Figure 3.

In this measurement method spatial resolution gets better for OPED than for FBP. But if the filter parameter is increased to 1.5 the curves of the MTF of FBP and OPED become comparable. In order to overcome aliasing due to the finite sampling in digital imaging system, oversampling of the LSF is typically employed [7]. Consequently, accurate data concerning SNR and spatial resolution could only be read out of aliasing-free oversampled LSF respectively MTF. Nonetheless OPED tends to result for both wire positions in an optimum, because it returns an interpolation polynomial, whose quadrature approximates the object function the most (see 2.3).

### 3.2 Simulation using synthetic data

To study the numerical stability we used for the following test synthetic data assuming a punctiform focal spot and no artefacts due to scatter etc.

For the test we used a model of a cylinder with a diameter of 40 mm and a fan angle of approximately 10°. Again, we measured the spatial resolution and the results are shown in Table 3. This time the OPED performs better compared to FBP as well for the SNR values as for the spatial resolution.

Table 3 shows that the SNR values of OPED and FBP are very high compared to the values in the test with the real data (see Table 2). Especially for OPED the SNR is nearly five times higher than for FBP. But on the other hand the spatial resolution of FBP is much smaller.
Table 3. Simulation: SNR and Spatial Resolution comparison of FBP and OPED

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Filter parameter</th>
<th>SNR</th>
<th>Spatial Resolution [Lp/mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>FBP</td>
<td>0.5</td>
<td>4703</td>
<td>0.68 ± 0.005</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>4153</td>
<td>1.38 ± 0.04</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>4038</td>
<td>1.99 ± 0.36</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>3963</td>
<td>2.24 ± 0.33</td>
</tr>
<tr>
<td>OPED</td>
<td>none</td>
<td>19497</td>
<td>3.37 ± 0.38</td>
</tr>
</tbody>
</table>

To study this effect in more detail we took a line profile of the tomogram reconstructed by FBP and OPED as shown in Figure 4 and Figure 5. The line profiles in Figure 5 show the typical gray-value curves for FBP and OPED along the dedicated cutting lines.

![Figure 4. Reconstructed cylinder with cutting line. Left: FBP, right: OPED](image)

The periodical oscillations of OPED’s line profile are not happened coincidentally. They depict the interpolation polynomial that oscillates around the ideal constant value inside the reconstructed cylinder. The Gibbs like artefacts near the sharp edges can be reduced, but this is outside the scope of our article.

The line profile of the FBP shows a cupping artefact due to a strong attenuation of the low spatial frequencies near the reconstruction center, i.e. in the center of the tomogram. This effect will lead to visible artefacts and explains the bad SNR compared to the OPED. The high oscillations at the edges of the cylinder have bad influence to the computation of the MTF, i.e. the discrete computation of the spatial resolution.

Anyway such obvious oscillations due to sharp edges as in this test with synthetic data will not appear in reconstructed tomograms of real data, because of the physical limits, which for example lead to artefacts like scatter or beam hardening and avoid sharp edges in real life. This test compares the numerical stability of both algorithms, i.e. it shows the advantage of the OPED algorithm (see 2.3).
4. Conclusions

In this article we introduced the orthogonal polynomial expansion on the disk (OPED) algorithm as a CT reconstruction algorithm well suited for fan-beam geometry. Nonetheless the OPED can be adapted to cone-beam geometry as well.

Today the state-of-the-art reconstruction method is the filtered backprojection (FBP), which has proven itself for many years. To motivate the need of another algorithm we compared the OPED and FBP in practical tests.

First we measured the computational effort. We showed that the OPED is always faster for the same large input data and the dedicated large reconstruction grids.

In another comparison we measured the spatial resolution. It came out that for real data the OPED has a 5% better spatial resolution than the FBP concerning their default settings, but the SNR was not comparable to each other.

Though adjusting the filter parameter of the FBP to a comparable SNR level, so that the spatial resolution becomes even higher than for the OPED, the standard deviation of the spatial resolution of the FBP increases also, i.e. the OPED tends to be more stable concerning the visual image quality.

To prove this effect we measured in the third test the PSF to get rid of superimposing artefacts like beam hardening or scatter for example. In this case the OPED performed better than the FBP.

The last test was done using synthetic data. The results verify the better numerical stability of the OPED algorithm.
In summary the OPED is faster regarding the computation time due to a more effective reordering of the fan-beam input data and the OPED is analytical exact, because the interpolating polynomial is the best among all polynomials of the corresponding degree approximating the object function in the sense of squared deviation.

Therefore the presented OPED algorithm will lead always to optimal results without wasteful analysis and adjustment of reconstruction parameters.

In this sense OPED makes the reconstruction process easier and faster, not only for large CT-volumes.

References


