Modeling of Ultrasonic Nondestructive Testing of Surface-breaking Cracks

Per-Åke JANSSON, Anders BOSTRÖM
Department of Applied Mechanics, Chalmers University of Technology,
Göteborg, Sweden; Phone: +46 31 7721526, Fax: +46 31 7723827;
e-mail: per-ake.jansson@chalmers.se, anders.bostrom@chalmers.se

Abstract
A complete model of ultrasonic nondestructive testing of defects in thick-walled components has been developed since more than a decade and has resulted in the computer program UTDefect. The program includes models of transmitting ultrasonic probes, and through a reciprocity argument also receiving probes. A number of simply shaped defects are included. Here the focus is on surface-breaking strip-like cracks, and a comparison of these with strip-like interior cracks. Both scattering problems are solved with a hyper-singular integral equation method, where the unknown crack-opening displacement (COD) is expanded into sets of Chebyshev functions. These expansions are different for the two cracks; whereas the interior crack has a COD that behaves as a square root at the tips, the surface-breaking COD has a finite value at the crack mouth, and these behaviours must be taken into account. Numerical comparisons are made between the two cracks, and it seems that the interior crack pushed to the surface behaves very much like the surface-breaking one, except at low frequencies (crack size less than half a wavelength). Favourable comparisons with experiments are also made.

Keywords: ultrasound, surface-breaking crack, integral equation.

1. Introduction

The scattering by cracks is an important problem in elastodynamics with applications particularly in ultrasonic nondestructive testing (NDT). The most common approaches include the finite element method (FEM), approximate high frequency methods (the geometrical theory of diffraction and Kirchhoff theory), and various integral equation methods. The cracks of interest in NDT are typically a couple of wavelengths across and the wave propagation volume may be 10–100 wavelengths across. In 3D this often gives excessively many unknowns for FEM. The high frequency methods, on the other hand, give efficient computations, but have an accuracy that is difficult to assess.

The integral equation approach to crack scattering yields hypersingular integrals which need some care to handle. Martin and Rizzo [1] give a review of the literature and discuss the origins of the hypersingularity and various ways to tackle it. One way to discretize the integral equation is to apply a boundary element approach (BEM). The singularity at the crack edge should then be included in some way. Another, more analytical, approach to the discretization of the integral equation is to use an expansion in some global (on the crack) system. This is only possible for simple crack shapes. In 3D this approach is used by Krenk and Schmidt [2] and Martin and Wickham [3] for the circular crack and by Guan and Norris [4] for the rectangular crack.

Surface-breaking and near-surface cracks are often critical in ultrasonic NDT and have therefore attracted attention. Recent studies, including also comparisons with experiments, are reported by Raillon et al. [5], who employ the Kirchhoff approximation, and Jansson and Boström [6], who draw on earlier work by Bövik and Boström [7] for a strip-like near-surface crack. In the latter work the problem is treated in 3D with a realistic model of an ultrasonic probe as transmitter and the electric signal in the receiving probe given by an electromechanical reciprocity relation. As a special case the crack is allowed to approach the surface and become surface-breaking. Both for internal and surface-breaking cracks comparisons with experiments yield reasonable results. However, the approach to the surface can be questioned as it gives a crack opening displacement...
(COD) that vanishes at the crack mouth as opposed to a finite value that is expected. In this paper this difficulty with the surface-breaking crack is avoided by choosing a new COD expansion that satisfies the square root behaviour at the internal crack edge and has a finite value at the crack mouth. Otherwise the present paper is very similar to Bövik and Boström [7], so for details not presented here that paper should be consulted.

2. Problem formulation

Consider a geometry with a surface-breaking strip-like crack of height $a$, which for simplicity is assumed to be normal to the surface of the half-space. A coordinate system $xyz$ is introduced with the $x$ axis normal to the crack, the $y$ axis along the crack mouth and the $z$ axis across the crack pointing into the half-space, see figure 1. The material in the half-space is homogeneous, isotropic, and linearly elastic with Lamé constants $\lambda$ and $\mu$ and density $\rho$. Time harmonic conditions are assumed with the time factor $e^{-i\omega t}$ suppressed. The longitudinal and transverse wave numbers are $k_p$ and $k_s$, respectively.

![Figure 1](image-url). The geometry of a plate with a surface-breaking crack.

To model a typical situation in ultrasonic NDT the surface-breaking crack is inserted into a plate geometry, with the crack situated at one side and an ultrasonic contact probe operating in pulse-echo at the other. For the modelling of the ultrasonic probe in transmission and reception, see Bövik and Boström [7]. As the crack scattering is solved in a half-space geometry this means that the multiple scattering between the crack and the scanning surface is neglected (or can be gated out in time). This is possible in many cases of practical importance, the main requirement being that the thickness of the plate is at least a couple of wavelengths.

3. The integral equation

Starting from the integral representation for the displacement field $u$ in the elastic half-space (see Ström [8]), operating with the traction operator and applying the stress-free boundary conditions on the crack and the half-space, the following integral equation is obtained

$$\lim_{x \to 0^-} \frac{k_s}{\mu} \int_{-\infty}^{\infty} dy' \int_0^a dz' \Delta u(r') \cdot \Sigma(r', r) = -t^1(r).$$

Here $\Delta u$ is the COD and $t^1$ is the traction of the incident field on the crack. The incident field is assumed to satisfy the traction-free boundary condition on the surface of the half-space. As the
Green tensor is also assumed to satisfy this boundary condition there only remains an integral over the crack. Furthermore, \( \Sigma \) is the Green stress tensor for the half-space that is obtained from the Green tensor by applying the traction operator to both variables of the Green tensor. The limit in front of the integrals cannot be taken inside the integrations as the integrals are hypersingular. The way the integral equation is solved in a natural way leads to regular integrals so the limit can at a later stage be taken inside the integrals.

The integral equation is solved by expanding the Green stress tensor in plane waves (a double Fourier transform) and the COD in Chebyshev functions. The expansion of the Green stress tensor is

\[
\Sigma_{n'n}(r', r) = 2i\mu^2 \sum_j \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dq dp}{k_j h_j} F_{jn'} F_{jn}^* e^{i(h_j|x'|+p(y' + q(y-z))}
+ 2i\mu^2 \sum_{j'j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dq dp}{k_j h_j} G_{jn'} R_{j'j} G_{jn}^* e^{i(q(x'-x)+p(y'-y)+h_jz'+h_jz)}.
\]  

(2)

Here \( n \) and \( n' \) run through 1, 2, 3 and \( j \) and \( j' \) are summed over 1, 2, 3. The wave numbers are \( k_1 = k_2 = k_n \) and \( k_3 = k_n \) and \( h_j = (k_j^2 - q^2 - p^2)^{1/2} \) with \( \text{Im} h_j \geq 0 \). The stress components of the plane waves \( F_{jn'} \) and \( G_{jn'} \) and the reflection matrix \( R_{j'j} \) are given in Bövik and Boström [7]. The first term in equation (2) is the free space Green stress tensor which has been Fourier represented in the tangential coordinates to the crack which means that the awkward term \( | x' - x | \) in the exponential vanishes when both coordinates are on the crack, i.e. when \( x = x' = 0 \). The second term is the half-space part which is Fourier represented in the tangential coordinates to the surface of the half-space, this being the only possibility.

The COD is expanded as

\[
\Delta u_{m'}(y', z') = \sum_{m'} \int_{-\infty}^{\infty} \frac{dp'}{k_a} \beta_{n'm'}(p') \phi_{m'}(z') e^{-p'y'},
\]

(3)

where \( m' \) is summed over all positive integers. The expansion functions are

\[
\phi_m(z) = \cos ((2m-1) \arcsin (z/a)), \quad m = 1, 2, 3, \ldots.
\]

(4)

These functions form a complete orthogonal set on \( z \in [0, a] \). They have a finite value at \( z = 0 \) and behave as \( \sqrt{a-z} \) when \( z \to a \), exactly as the COD must behave. When the ansatz (3) and the Green expansion (2) are inserted into the integral equation (1) the following integral appears

\[
I_m(\gamma) = \int_0^1 \cos ((2m-1) \arcsin(t)) e^{i\gamma t} dt.
\]

(5)

Unfortunately, it seems that this integral cannot be calculated in simple analytical form, but except for large \( \gamma \) it is straightforward to compute fast numerically. It is noted that the corresponding integral which appears for an interior crack gives essentially a Bessel function, exactly as the real part of equation (5) for real \( \gamma \) gives a Bessel function. The properties of \( I_m(\gamma) \) are further explored in the Appendix.

Inserting equations (2) and (3) into equation (1), projecting with the expansion system (4) and taking the Fourier transform with respect to \( y \) gives a discretized form of the integral equation:

\[
\sum_{n'm'} Q_{nn'm'}(p) \beta_{n'm'}(p) = \frac{1}{2\pi \mu a} \int_{-\infty}^{\infty} dy \int_0^a dz t_n(y, z) \phi_m(z) e^{ipy}.
\]

(6)
Here the system matrix is

\[ Q_{nnm'm'}(p) = -4\pi i a \int_{-\infty}^{\infty} dq \left( \sum_j \frac{1}{k_j h_j} F_{jn'} F_{jn}^* I_m(q a) I_{m'}(-q a) \right. \]

\[ + \left. \sum_{jj'} \frac{1}{k_{jj'} h_{jj'}} G_{jn'} R_{jj'} G_{jn}^* I_m(h_{jj'} a) I_{m'}(h_{jj'} a) \right). \]  

(7)

In the discretized version equation (6) of the integral equation the limit in front of the integrals in the original integral equation (1) has disappeared. This is possible because the \( q \) integral in equation (7) is convergent in the ordinary sense. This is not at all obvious and is therefore clarified in the Appendix.

Solving equation (6) for the expansion coefficients \( \beta_{nm} \) and substituting into equation (3) yield the COD. It is then straightforward to obtain the change in the recorded signal by using an electromechanical reciprocity argument by Auld [10]:

\[ \delta \Gamma = -\frac{i \omega}{4P} \int_{-\infty}^{\infty} \int_0^a \Delta u_j(y, z) \sigma_{1j}(y, z) dy \, dz, \]  

(8)

where \( P \) is the incident electric power to the probe, \( \Delta u_j \) is the COD due to the incoming field, and \( \sigma_{1j} \) is the traction in the absence of the crack. Here the same probe model in both transmission and reception as used by Bövik and Boström [7] is used, and all details can thus be found there and are not repeated here. The quantity \( \delta \Gamma \) denotes the extra electric reflection coefficient from the receiving probe due to the presence of the crack; this is essentially the quantity measured in practice. After some algebra the signal response may be expressed as

\[ \delta \Gamma = \int_{-\infty}^{\infty} T_{nm}(p) Q_{nm'm'}^{-1}(p) T_{n'm'}(p) \, dp. \]  

(9)

The matrix \( T \) basically depends on the field generated by the transmitter and has the following form:

\[ T_{nm}(p) = \int_{-\infty}^{\infty} \sum_{j=1}^{3} L_{nmj}(p, q) e^{i(-qd_z + h_j d_z)} \, dq. \]

(10)

The expression for \( L_{nmj} \) is not given here, but is in principle given for a crack with a closed crack mouth in [7]. For an open crack mouth the main difference is that a Bessel function needs to be replaced by the function \( I_m \) defined by equation (5). If the crack is located in the far field of the transmitter the \( q \)- and \( p \)-integrals in equations (9) and (10) can be calculated approximately using the stationary-phase approximation, which will speed up the computation of the signal response considerably compared to a numerical integration.

4. Numerical results

There are numerous parameters that can be varied in this problem, but in the present study the emphasis is on the effect of the condition at the crack mouth, and the validity of the stationary-phase approximation.

In figures 2 and 3 the effect of using the more realistic boundary condition with a finite COD is illustrated. The longitudinal and shear wave speeds are 5.9 and 3.2 mm/\( \mu \)s, respectively, and the thickness of the plate (\( d_z \)) is 30 mm. The transmitter is a 45° SV probe with a circular contact area of 20 mm diameter operating in pulse-echo mode at a fixed frequency of 2 MHz. From experience it is known that the signal response for this case is not likely to differ considerably from that of
a probe operating in the time domain with the same centre frequency for the spectrum. It should be noted that this frequency corresponds to a shear wavelength of 1.6 mm. The integrals in equations (9) and (10) are computed numerically. For all cases the results are calibrated against a side-drilled hole of 2 mm diameter at a depth of 30 mm.

In figure 2 the height of the crack is 2 mm, and both models give very similar results. In figure 3 on the other hand, where the crack width is 0.5 mm, there is a difference of more than 5 dB between the maximum values. Obviously, it is important to use the correct crack mouth condition when the size of the crack is smaller than the wavelength. This seems like a reasonable result as the effect of diffraction is expected to be dominant here, while mirror reflection is more important for the 2 mm crack.

The open mouth model has also been validated against some benchmark problems that were published by the World Federation of NDE Centers in 2009 [11]. In the experiments scattering from a rectangular surface-breaking crack of 40 mm length was studied for various values of the height of the crack. Here the finite length crack is approximated by a strip-like crack, and the signal response is calculated using the stationary-phase method as well as numerical integration of the \( p \) - and \( q \)-integrals. To increase the accuracy of the stationary-phase approximation, the area of the transmitter is divided into smaller elements to ensure that the crack is in the far field of the transmitter. As before the response is calculated at a fixed frequency, the centre frequency for the frequency spectrum of the probe used in the experiments. For more details of the experiments, the reader is referred to [11].

The results are presented in Table 1 for various values of the height of the crack. All results are given in dB calibrated against a side-drilled hole. It is seen that the agreement with the experimental results is excellent when the integrals are computed numerically. For the smaller cracks the stationary-phase approximation also yields results in very good agreement with experiments.

![Figure 2](image1.png)

**Figure 2.** The signal response from a 2 mm surface-breaking crack as a function of the position of the transmitter using a closed (solid curve) and an open (dashed curve) crack mouth.

![Figure 3](image2.png)

**Figure 3.** The signal response from a 0.5 mm surface-breaking crack as a function of the position of the transmitter using a closed (solid curve) and an open (dashed curve) crack mouth.
Table 1. Comparison between experimental results for rectangular cracks of length 40 mm and numerical simulation for strip-like cracks.

<table>
<thead>
<tr>
<th>Height (mm)</th>
<th>Experimental</th>
<th>Simulation (numerical integration)</th>
<th>Simulation (stationary-phase method)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>8.0</td>
<td>7.9</td>
<td>6.9</td>
</tr>
<tr>
<td>5</td>
<td>13.3</td>
<td>12.7</td>
<td>13.4</td>
</tr>
<tr>
<td>10</td>
<td>13.7</td>
<td>13.7</td>
<td>20.4</td>
</tr>
<tr>
<td>20</td>
<td>13.5</td>
<td>13.0</td>
<td>25.9</td>
</tr>
</tbody>
</table>

For the larger cracks, however, the simulations give a much stronger response. A possible explanation for this is that even though the crack is in the far field of the probe, the distance between the probe and the crack is not large compared to the near-field length of the crack (defined in analogy with the near-field length of a probe).

5. Conclusions

An investigation of the surface-breaking strip-like crack, and in particular the behaviour at the crack mouth and the use of the stationary-phase approximation, has been performed. A comparison with an interior crack that is pushed to the surface shows that the crack mouth behaviour is important only for small cracks, approximately for cracks less than a wavelength. Use of the stationary-phase approximation yields good results in many cases, but care must be taken so that the conditions for its validity are not violated. This includes the condition that the distance between the crack and the probe must be large, both in terms of the near-field lengths of the probe and the crack, and in terms of the wavelengths. Comparisons with experiments yield excellent results, in spite of the fact that only a fixed frequency is used.

Acknowledgement

The present work is sponsored by the Swedish Radiation Safety Authority (SSM) and this is gratefully acknowledged.

Appendix

In the discretized version, equation (6), of the integral equation the limit in front of the integrals in the original integral equation (1) has disappeared. This is possible because the \( q \) integral in equation (7) is convergent in the ordinary sense. This is not at all obvious and is therefore clarified in this appendix.

First the asymptotic behaviour for large arguments of the function \( I_m(\gamma) \) defined in equation (5) must be investigated. For a purely imaginary argument (which is relevant for the second term in the integrand in equation (7)) \( \gamma = is \), \( s \) real, the definition is

\[
I_m(is) = \int_0^1 \cos((2m - 1)\arcsin t)e^{-st}dt.
\]  

A repeated integration by parts (Olver [9], note that only the lower limit contributes) gives

\[
I_m(is) = \frac{1}{s} + O(s^{-3}), \quad (12)
\]
independently of the order \( m \).

For a purely real argument \( \gamma \) the situation is a little more complicated. Start with a partial integration and a change of variable to \( x = 1 - t \) to obtain

\[
I_m(\gamma) = \frac{i}{\gamma} - \frac{i}{\gamma} e^{i\gamma} \int_0^1 e^{-i\gamma x} \sin \left((2m - 1) \arcsin(1 - x)\right) \frac{2m - 1}{\sqrt{2x - x^2}} \, dx. \tag{13}
\]

The dominating contribution here comes from the lower limit. Using a theorem by Olver [9] (page 101) gives

\[
I_m(\gamma) = \frac{i}{\gamma} + \frac{1}{\gamma^{3/2}} \sqrt{\frac{\pi}{2}} (-1)^m (2m - 1) e^{i(\gamma + \pi/4)} + O(\gamma^{-5/2}). \tag{14}
\]

Another way to obtain the real part of the second term is to realize that the real part of \( I_m(\gamma) \) for real arguments is essentially a Bessel function and to use the well-known asymptotic behaviour of the Bessel function.

The other factors in equation (7) are straightforward to estimate asymptotically. Thus the largest elements of \( F_{jn}, G_{jn} \) and \( R_{jj'} \) all behave as \( O(q^2) \). This seems to lead to strongly diverging integrals, but due to cancellations in the summations this is not so. The dominating terms in the two summations are in fact

\[
16\pi^2 \sum_j \frac{1}{k_j h_j} F_{jn'} F_{jn}^* = \delta_{nn'} \alpha_n \ | \ q \ | + O(1), \tag{15}
\]

\[
16\pi^2 \sum_{jj'} \frac{1}{k_j h_j} G_{jn'} R_{jj'} G_{jn}^* = \delta_{nn'} \alpha_n \ | \ q \ | + O(1), \tag{16}
\]

where \( \alpha_1 = \alpha_3 = 2i(k^2_s - k^2_p) \) and \( \alpha_2 = i \). The \( O(1) \) terms only appear for \( n \neq n' \), for \( n = n' \) the next order terms are \( O(q^{-1}) \). Remembering the behaviour of \( I_m \) given in equations (12) and (14) it is seen that the two terms in equation (7) are still divergent as the integrands behave as \( O(q^{-1}) \). However, these leading order terms cancel because the same factor \( \alpha_n \) appears in both equations (15) and (16). This is natural because these cancelling terms can be traced to singularities due to the crack mouth behaviour which cancel in the stress Green tensor due to the stress free boundary condition.

For \( n = n' \) the remaining leading order terms in the integrand in equation (7) all come from the first term. The dominating terms for \( q \to \infty \) can be rewritten as

\[
\frac{i\alpha}{a(qa)^{3/2}} \sqrt{\frac{\pi}{2}} \left[(\beta_{m'} - \beta_m) \cos (qa + \pi/4) - i(\beta_{m'} + \beta_m) \sin (qa + \pi/4)\right] + \frac{\pi \alpha_n \beta_m \beta_{m'}}{2a^3 q^2}, \tag{17}
\]

where \( \beta_m = (-1)^m (2m - 1) \). For \( q \to -\infty \) the integrand is even except that \( m \) and \( m' \) have to be exchanged. Using the asymptotic form of the Bessel functions \( J_1 \) and \( J_2 \) the integrand can thus to leading orders be rewritten as

\[
\frac{i\alpha_n}{a^2 |q|} \left[(\beta_{m} - \beta_{m'}) \beta_{m'} J_1(qa) + i(\beta_m + \beta_{m'}) J_2(qa)\right] + \frac{\pi \alpha_n \beta_m \beta_{m'}}{2a^3 q^2}, \tag{18}
\]

which is valid for both positive and negative \( q \). The last term is even and as it can be integrated analytically over arbitrary intervals the contributions from the "tails" outside the truncation limits can be added. The \( J_1 \) term is odd so this term can simply be subtracted from the integrand. The
The $J_2$ term is even and can be subtracted and added to the integrand with the added part integrated analytically with the help of the following integral:

$$\int_0^{\infty} J_\nu(\beta x) \frac{dx}{x} = \frac{1}{\nu},$$  \hspace{1cm} (19)$$

which is valid for all $\nu > 0$. With this the integrand decays as $O(q^{-5/2})$ which can be integrated numerically with reasonable accuracy.

For $n \neq n'$ the dominating parts in the integrand behave as $O(q^{-2})$ and these can be treated as the last term in equation (18).

References