Mathematical Modeling of Radiography Experiments

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Abstract
An approach is presented to construct operators for transforming the characteristics of incident radiation to transmitted radiation, as well as operators for transforming the transmitted radiation to measured values. Simulating the radiation transport is based on Monte Carlo modeling of the interaction of X-ray photons and electrons with matter. The proposed method permits to construct, for instance, the operator connecting the initial radiation spectrum with the absorbed photon energy penetrating a given object. The elaborated approach provides the possibility of effective mathematical modeling of radiation techniques such as radiography, treating complex multi-component objects. Moreover, the method can be used to construct the operator equation for solving inverse problems, e.g. the reconstruction of the initial radiation spectrum using simple experimental measurements. Comparison with some experimental measurements is presented.

Keywords
radiography, operator of radiation transformation, mathematical modeling, spectrum reconstruction

1. Introduction

Mathematical modeling is an effective and powerful means for radiography. It is useful for developing complex experimental methods [1], as well as investigation of effectiveness and reliability of technical equipment [2]. For instance, determining the range of application of radiation sources is an important problem in NDE. It requires performing a large number of calculations using complex methods of radiation transport modeling. The developed approach for constructing the operators of transformation of the incident X-ray spectrum into registered radiation allows carrying out a series of very simple calculations (multiplying a matrix by a column vector) for a number of sources with different spectra. Design of experiments (DoE), optimization of NDT process, and development of innovative X-ray sources all require correct information about X-ray source spectra. Determining radiation spectra is a necessary part of experimental methods in NDE. Some known methods of spectra determination are based on measurement of a function of the wanted spectrum and following mathematical treatment of the measurement results [3, 4]. With this approach an important task is the construction of an operator equation connecting incident spectrum and measured values. An effective technique of operator construction is presented in this paper. The technique is based on calculating a discrete approximation to the transformation operator in question. Some examples are presented.
2. Technique for construction of spectrum transformation operators

2.1 Transformation of spectra

Let us consider the general scheme of an NDE experiment (fig. 1). The incident radiation with spectrum \( f(E) \) propagates through an object and transforms into radiation with a different spectrum \( \varphi(E) \) behind the object. Finally the radiation with spectrum \( \varphi(E) \) is registered by a detector system. The energy distribution of measured values is indicated as \( F(E) \). Let us identify the transformation \( f(E) \rightarrow \varphi(E) \) as \( \varphi(E) = Af(E) \) and \( \varphi(E) \rightarrow F(E) \) as \( F(E) = P \varphi(E) \), \( f, \varphi, F \in U \), where \( U \) is the functional space of generalized functions.

We assume that the radiation transport through the object does not change properties of the object matter and particles of the radiation do not interact with each other. In that case \( A, P \) are linear operators.

![Fig. 1. General scheme of NDE experiment.](image)

\[ A, P \]

2.2 Construction of operator \( A \)

Constructing the operators and analyzing their properties implies modeling of radiation transport processes and building the functional corresponding to the measured values. The technique for the transformation operator \( f(E) \rightarrow \varphi(E) \) (see fig. 1) is described below.

Let us consider \( f_\delta(E) = \delta(E - E') \) as the spectrum of a mono-energetic source. We introduce the function \( G(E, E') \) as result of applying operator \( A \) to \( f_\delta(E) \):

\[ G(E, E') = A \delta(E - E'). \]

In that case the function \( G(E, E') \) is the energy density distribution of registered radiation for a mono-energetic source with energy \( E' \). This function is called the Green function of the transformation \( f(E) \rightarrow \varphi(E) \) or the Green function of the experiment [5].

It is not difficult to obtain the transmitted spectrum from the Green function for any \( f(E) \):

\[ Af(E) = A \int f(E') \delta(E - E') dE' = \int f(E') A \delta(E - E') dE' \], and finally

\[ \varphi(E) = Af = \int f(E') G(E, E') dE'. \] (1)
Eq. (1) shows that the spectrum of registered radiation $\varphi(E)$ can simply be found as the convolution of the Green function $G(E, E')$ and $f(E)$ for any energetic distribution of incident radiation $f(E)$.

Thus the construction of operator $A$ is reduced to finding the function $G(E, E')$. Determining the Green function is a nontrivial task, requiring taking into account the concrete physical conditions of an experiment. The general approach to construction of $G(E, E')$ is mathematical modeling of photon transport by use of effective calculating schemes (see for instance [6]). This scheme is based on carrying out the statistical estimation of a discrete approximation of $G(E, E')$ and using this estimate for subsequent analysis and investigation.

Let us consider one simple example where $G(E, E')$ can be determined analytically. Let $S$ be a plane X-ray source of photons with a spectrum $f(E)$ incident normally on a plate of thickness $d$ and macroscopic total interaction cross-section $\mu$. The number of incident photons is $dn = f(E)dE$. The number of non-scattered photons propagating through the plate is $dn' = \varphi(E)dE$. For $dn'$ follows:

$$dn'(E') = dn \cdot \exp\left(-\mu(E)d\right) \delta(E - E').$$

This implies $\varphi(E) = \int f(E')\exp\left(-\mu(E')d\right) \delta(E - E')dE'$. Therefore the kernel of $A$ (Green function) is $G(E, E') = \exp\left(-\mu(E')d\right) \delta(E - E')$ and the spectrum of transmitted photons without scattering, taking into account eq. (1), is $\varphi(E) = Af = f(E)\exp\left(-\mu(E)d\right)$.

Analytical determination of the Green function $G(E, E')$ is impossible in most practical situations. In this case $G(E, E')$ is calculated by means of mathematical modeling. Two examples are given below.

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Fig. 2. Example of operator $A$ for registration of transmitted photons.
Let us consider a model experiment. An aluminum plate of 1 mm thickness is normally irradiated by a pencil beam of X-ray with energy up to 100 keV. The Green function $G(E, E')$ for transformation of the incident spectrum into transmitted photons is presented in fig. 2. It was calculated using algorithms from [6] and the corresponding Monte Carlo code McRay. The image of the operator for the transformation of photon radiation into electron flux is presented in fig. 3. Calculations were carried out using KIAM code for modeling the electron emission due to photon interaction [7].

Fig. 3. Example of operator $A$ for registration of emitted electrons.

### 2.3 Construction of operator $P$

We now consider the transformation of registered radiation into a measured value $\varphi(E) \rightarrow F(E)$ (see fig. 1). The corresponding operator is $P: F(E) = P\varphi(E)$. Let $\varphi_\delta(E) = \delta(E - E')$. Specify $P\delta(E - E') = G_D(E, E')$. Function $G_D(E, E')$ is the response function of a detector, i.e. the Green function of the transformation $\varphi(E) \rightarrow F(E)$. Then we obtain:

$$P\varphi(E) = P\int \varphi(E')\delta(E - E')dE' = \int \varphi(E')P\delta(E - E')dE'. \quad (3)$$

We can write eq. (3) in the form $P\varphi(E) = \int \varphi(E')G_D(E, E')dE'$. Thus, for any $\varphi(E)$ follows $F(E) = \int \varphi(E')G_D(E, E')dE'$. Calculation of $G_D(E, E')$ for a given detector system requires taking into account all interaction processes between radiation and matter resulting in mechanisms of detection. For instance, if we consider a semiconductor detector based on Si the processes of photo absorption and Compton scattering should be modeled. Moreover we have to compute the transport of electrons produced by photons in the detector for correct determination of the energy deposit. An example for calculation of operator $P$ is presented in fig. 4. A Si detector is used. The scheme of the experiment is given on the left and an image of $G_D(E, E')$ is shown on the right. The Si detector of 0.3 mm thickness is irradiated by a plane X-ray source with energies...
from 1 keV up to 700 keV. The energy distribution of the absorbed radiation is measured. The structure of the shown function results from the joint influence on $G_D(E, E')$ of both Compton scattering and photo absorption in the detector material. Calculations were carried out using KIAM code for Monte Carlo modeling the photon detection.

3. Building an equation for spectra reconstruction

As mentioned above an important problem in NDT experiments is the determination of the radiation spectrum used. Here we present an analysis of a spectrum based on mathematical treatment of measured data. The main part of the treatment is the construction of an operator equation connecting the wanted spectrum and the measured values. The equations are then solved using a robust technique [8]. An approach for building the required integral equation is presented in this paper.

Let $F(E)$ be the energy distribution of the measured functional. We can write:

$$F(E) = P \varphi(E) = P A f(E). \quad (4)$$

Eq. (4) can be written in the form:

$$F(E) = \int dE' dE' G_D(E, E') G(E', E^*) f(E^*). \quad (5)$$

Specifying $G_p(E, E') = \int dE' G_D(E, E') G(E', E^*)$ we can rewrite eq. (5) as

$$F(E) = \int dE' G_p(E, E') f(E'). \quad (6)$$

Eq. (6) is an integral equation of the 1st kind for the determination of $f(E)$.

Let us consider an example experiment for determination of the spectrum of incident radiation (X-rays of 110 keV maximum energy from a 45° tungsten target). The scheme of the experiment is presented in fig. 5. X-ray photons fall on an aluminum plate of 3.5 mm thickness and the transformed radiation after photon transmission is measured by a Si detector. Electrons emitted from the aluminum plate are considered in the measurement process.
The KIAM codes for Monte Carlo modeling the photon transport and photon detection were used for computing the function $G_p(E)$ (see [4-6]).

The KIAM code for solving ill-posed integral equations of the 1st kind was applied for determination of the incident radiation spectrum [8]. This code realizes the corresponding algorithm for solving an operator equation of the 1st kind. The algorithm is based on minimizing the variation of the second left derivative of the solution.

The result of the spectrum determination presented in fig. 6 demonstrates the applicability of the developed approach for solving the type of problem considered in this paper.
4. Conclusion

The developed technique presented in this paper is applicable for:

- Mathematically modeling the interaction of photons with objects/matter and the spectral registration of the photon radiation by detectors;
- Mathematically modeling the electron emission from photon interactions including spectral registration;
- Analyzing various experimental schemes for optimizing the conditions of experiments (Design of Experiments);
- Reconstructing the initial source spectrum from measurements for investigating the characteristics of modern and innovative radiation sources.

References