Combined Empirical and Analytical Virtual Sensing for Full-Field Dynamic Response Estimation

Jyrki KULLAA

Department of User-centered Design and Production,
Helsinki Metropolia University of Applied Sciences,
Helsinki, Finland
jyrki.kullaa@metropolia.fi

Abstract
A full-field dynamic response can be estimated using a limited number of vibration sensors on a structure. Virtual sensing techniques estimate unmeasured quantities using the available physical sensors. Combined empirical and analytical virtual sensing is introduced to reduce the uncertainty of the estimation. First, empirical virtual sensing utilizes the redundancy of the sensor network to compute less noisy signal estimates. The actual noisy measurements are then replaced with their empirical estimates in subsequent analytical virtual sensing. Analytical virtual sensing utilizes the mode shapes obtained from the finite element model as a basis to estimate the response of the unmeasured quantities of interest. Four different algorithms were compared using numerical simulations: analytical and combined virtual sensing using either ordinary least squares (OLS) or weighted least squares (WLS). Combined virtual sensing outperformed analytical virtual sensing in all cases.

Keywords: Virtual sensing, Structural dynamics, Bayesian analysis, Response estimation, Sensor network, Strain field estimation.

1. INTRODUCTION

Sensors and monitoring systems appear everywhere. Current technology makes it possible to install a wireless sensor network with a high number of small low-cost sensor nodes. An important application is structural monitoring, which utilizes vibration measurements acquired by a sensor network to assess the condition of the structure. For example, fatigue assessment uses stress or strain histories at critical locations of the structure. However, some locations may be impractical or inaccessible for instrumentation. Therefore, accurate estimation of the response at these locations using available measurements would make fatigue assessment more reliable.

Virtual sensing techniques estimate unmeasured quantities using the available physical sensors. In vibration monitoring, the response of linear structures can be assumed to consist of the sum of modal contributions, in which only a few natural modes are active. With this assumption, a finite number of sensors is sufficient to make the sensor network redundant. This redundancy can be utilized in the estimation of unmeasured quantities.

Virtual sensing can be either analytical (model-based) or empirical (data-driven) [1]. Analytical virtual sensing techniques use information available from a limited set of physical sensors together with a finite element model to calculate an estimate of the quantity of interest. For example, it is possible to estimate the stress or strain field from acceleration...
measurements. Analytical mode shapes from the finite element model can be used as a basis to estimate the response at unmeasured locations by an expansion algorithm [2].


Empirical virtual sensing can be used to replace a temporarily installed or failed sensor [12]. Empirical virtual sensing has also been used for damage detection or sensor fault identification [13]. The accuracy of estimation can be increased by increasing the number of sensors in the network.

In this paper, a combined virtual sensing algorithm is introduced, which applies both empirical and analytical virtual sensing to obtain an estimate of the quantity of interest. The algorithm uses output-only data, in which the structural response is only measured, while the excitation remains unknown. The objective of the combined method is to reduce uncertainty of virtual sensing techniques due to measurement error. A redundant sensor network is assumed. Although an exact FE model is unrealistic, model error is ignored in this paper, making it possible to isolate the measurement error as the sole source of error thus facilitating the comparison of different algorithms. Also, errors due to double integration of accelerations are omitted for the same reason by presuming direct displacement measurements.

The paper is organized as follows. Empirical virtual sensing and analytical virtual sensing are introduced in Section 2 and a combination of both techniques for more accurate virtual sensors is proposed. A numerical study is performed in Section 3 with noisy vibration measurements to validate the proposed method and to compare four different algorithms. Concluding remarks are given in Section 5.

2. VIRTUAL SENSING IN STRUCTURAL DYNAMICS

2.1 Empirical virtual sensing

Empirical virtual sensing is based on available current or historical measurements. Consider a sensor network measuring p simultaneously sampled variables \( y(t) \) at time instant \( t \). Each measurement \( y \) includes measurement error \( w \):

\[
y = x_m + w
\]

where \( x_m \) are the true values of the measured degrees of freedom (DOF). All vectors are divided into predicted DOFs \( u \) and the remaining DOFs \( v \):
\[
\begin{align*}
\mathbf{y} &= \begin{bmatrix} y_u \\ y_v \end{bmatrix}, \\
\mathbf{x}_m &= \begin{bmatrix} x_{m,u} \\ x_{m,v} \end{bmatrix}, \\
\mathbf{w} &= \begin{bmatrix} w_u \\ w_v \end{bmatrix}
\end{align*}
\] (2)

For simplicity but without loss of generality, assume zero-mean variables \( \mathbf{y} \). The partitioned data covariance matrix \( \Sigma_y \) is

\[
\Sigma_y = \mathbb{E}[\mathbf{yy}^\top] = \begin{bmatrix}
\Sigma_{y,uu} & \Sigma_{y,uv} \\
\Sigma_{y,uv} & \Sigma_{y,vv}
\end{bmatrix} = \begin{bmatrix}
\Gamma_{y,uu} & \Gamma_{y,uv} \\
\Gamma_{y,uv} & \Gamma_{y,vv}
\end{bmatrix}^{-1} = \Gamma_y^{-1}
\] (3)

where the precision matrix \( \Gamma_y \) is defined as the inverse of the covariance matrix \( \Sigma_y \) and is also written in partitioned form. \( \mathbb{E}(\cdot) \) denotes the expectation operator.

A linear MMSE estimate for \( y_u \mid y_v \) \((y_u \text{ given } y_v)\) is obtained by minimizing the mean-square error (MSE) and can be computed either using the covariance or precision matrix [13, 14]. The expected value of the predicted variable is:

\[
\hat{y}_u = \mathbb{E}(y_u \mid y_v) = -\Gamma_{y,uu}\Gamma_{y,uv}y_v = \mathbf{K}y_v
\] (4)

Measurement error \( \mathbf{w} \) is assumed to be zero mean Gaussian, independent of \( \mathbf{x}_m \), with a (known) covariance matrix

\[
\Sigma_w = \mathbb{E}[\mathbf{ww}^\top] = \begin{bmatrix}
\Sigma_{w,uu} & \Sigma_{w,uv} \\
\Sigma_{w,uv} & \Sigma_{w,vv}
\end{bmatrix}
\] (5)

The objective is to find a better estimate for \( x_{m,u} \) than the actual measurement \( y_u \) utilizing the noisy measurements \( \mathbf{y} \) from the sensor network. From Bayes’ rule,

\[
p(x_{m,u} \mid y_u, y_v) = \frac{p(y_u \mid x_{m,u}, y_v) p(x_{m,u} \mid y_v)}{p(y_u \mid y_v)}
\] (6)

where the likelihood is

\[
p(y_u \mid x_{m,u}, y_v) = p(y_u \mid x_{m,u}) = \mathcal{N}(y_u \mid x_{m,u}, \Sigma_{w,uu})
\] (7)

and the prior is

\[
p(x_{m,u} \mid y_v) = \mathcal{N}(x_{m,u} \mid \mathbf{Ky}_v, \Sigma_{\text{prior}})
\] (8)

where \( \mathbf{K} = -\Gamma_{y,uu}\Gamma_{y,uv} \), and the prior covariance is

\[
\Sigma_{\text{prior}} = \Gamma_{y,uu}^{-1} - \Sigma_w
\] (9)

which were derived using (1) and the linear MMSE estimate for \( y_u \mid y_v \) [13, 14]. The denominator \( p(y_u \mid y_v) \) in (6) is the normalizing factor, which does not depend on \( x_{m,u} \).

The posterior distribution (6) is obtained by some manipulation, resulting in
where the posterior covariance \( \Sigma_{\text{post}} \) is
\[
\Sigma_{\text{post}} = \text{cov}(x_{m,u} \mid y_u, y_v) = (\Sigma_{w,u}^{-1} \Sigma_{\text{prior}}^{-1})^{-1}
\] (11)
and the posterior mean is
\[
\hat{x}_{m,u} = E(x_{m,u} \mid y_u, y_v) = \Sigma_{\text{post}}^{-1} (\Sigma_{w,u}^{-1} y_u + \Sigma_{\text{prior}}^{-1} K y_v)
\] (12)

Notice that the posterior mean (12) is a weighted average of the noisy measurement and the MMSE estimate.

Equation (12) can also be written in the following matrix form.
\[
\hat{x}_{m,u} = E(x_{m,u} \mid y_u, y_v) = [\Sigma_{\text{post}}^{-1} \Sigma_{w,u}^{-1} \Sigma_{\text{post}}^{-1} \Sigma_{\text{prior}}^{-1} K] \begin{bmatrix} y_u \\ y_v \end{bmatrix} = a_u^T y
\] (13)

where the coefficient row vector \( a_u^T \) is
\[
a_u^T = [\Sigma_{\text{post}}^{-1} \Sigma_{w,u}^{-1} \Sigma_{\text{post}}^{-1} \Sigma_{\text{prior}}^{-1} K]
\] (14)

For each sensor, a separate vector \( a_u^T \) is computed. All these vectors can be assembled in a coefficient matrix \( A \) to compute all estimates simultaneously:
\[
\hat{x}_m = A y
\] (15)
where each row of matrix \( A \) represents the corresponding sensor.

The estimated sensor signals (4) or (15) will be substituted for the noisy measurement in combined empirical and analytical VS as discussed in Section 2.3.

### 2.2 Analytical virtual sensing

Analytical virtual sensing uses a mathematical model of the system together with available measurements to estimate the quantity of interest. In this study, ordinary least squares and weighted least squares estimation are investigated.

In linear structural dynamics, the structural response can be written as a sum of modal contributions:
\[
x(t) = \sum_{i=1}^{N} \phi_i(t) q_i(t) = \sum_{i=1}^{n} \phi_i(t) q_i(t) = \Phi q(t)
\] (16)
where \( x(t) \) is the displacement response, \( N \) is the number of DOF in the finite element model, \( n << N \) is the selected number of active modes, \( \Phi \) is the truncated modal matrix consisting of the selected mode shapes \( \phi_i \) as columns, and \( q(t) \) are the modal, or generalized coordinates.
The System Equivalent Reduction Expansion Process (SEREP) algorithm [2] is briefly outlined. If \( x(t) \) is divided into measured and unmeasured DOF, \( x_m(t) \) and \( x_u(t) \), respectively, the mode shape vectors are divided correspondingly:

\[
x(t) = \begin{bmatrix} x_m(t) \\ x_u(t) \end{bmatrix} = \begin{bmatrix} \Phi_m \\ \Phi_u \end{bmatrix} q(t)
\]  

(17)

The upper equation reads

\[
x_m(t) = \Phi_m q(t)
\]  

(18)

If the number of sensors is greater than the number of active modes \( n \), the modal coordinates \( q(t) \) can be solved from (18) using ordinary least squares (OLS) solution

\[
\hat{q}(t) = (\Phi_m^T \Phi_m)^{-1} \Phi_m^T y(t)
\]  

(19)

or weighted least squares (WLS) solution:

\[
\hat{q}(t) = (\Phi_m^T W \Phi_m)^{-1} \Phi_m^T W y(t)
\]  

(20)

where the weighting matrix is \( W = \Sigma_u^{-1} \).

Once the estimate of \( q(t) \) is solved, all DOFs can be computed from (17):

\[
\hat{x}(t) = \Phi \hat{q}(t)
\]  

(21)

The mode shapes typically consist of displacement and rotation DOFs. They can also be augmented with strains or stresses at different locations of the structure, because the displacement pattern determines the strain or stress distribution.

2.3 Combined empirical and analytical virtual sensing

In the previous section, the noisy measurements \( y \) are used directly in analytical VS. With a high noise level, the virtual sensors may have large uncertainty. In combined virtual sensing, both empirical and analytical VS are used to obtain a more accurate estimate of the quantity of interest. First, empirical virtual sensing is used to estimate a less noisy measurement. The empirical estimate will then replace the actual noisy measurement \( y(t) \) in subsequent analytical VS in Equations 19 or 20.

Choosing the empirical estimates depends on whether OLS or WLS is used. OLS will be used if the noise covariance matrix is unknown, and therefore the weighting matrix \( W \) is not available. In that case, \( \hat{y}_u \) in Equation 4 is used as the empirical estimate. In WLS, the weighting matrix \( W \) is obtained using the diagonal elements of the posterior covariance matrix: \( W = \left[ \text{diag}(\Sigma_{\text{post}}) \right]^{-1} \), and \( \hat{x}_m(t) \) in Equation 15 is used to compute the empirical estimate.

3. PLANE FRAME

An experiment was performed with a numerical model of a two-dimensional steel frame (Figure 1) with a height of 4.0 m and a width of 3.0 m. Both columns were fixed at the bottom. The frame was also supported with a horizontal spring at an elevation of 2.75 m with
a spring constant of 2.0 MN/m. The frame was modelled with simple beam elements with square hollow section of 100 mm × 100 mm × 5 mm. The FE model consisted of 176 beam elements, 62.5 mm in length, and a single spring element. Nodes 1–177 and beam elements 1–176 were located sequentially starting from the bottom of the left column to the bottom of the right column.

Horizontal random loading was applied to the right column at nodes 113, 129, and 145, corresponding to elevations of 4 m, 3 m, and 2 m, respectively (Figure 1). The loads were mutually independent having standard deviations of 9 kN, 7 kN, and 5 kN, respectively. All load signals were low-pass filtered below 50 Hz.

Modal superposition was used to analyse the response of the structure. Seven lowest modes were included in the analysis with natural frequencies of 13.86 Hz, 37.10 Hz, 49.96 Hz, 59.40 Hz, 111.8 Hz, 119.3 Hz, and 166.7 Hz. Modal damping was assumed with damping ratios of $\zeta_1 = \zeta_2 = 0.01$, $\zeta_3 = 0.015$, and $\zeta_4 = \zeta_5 = \zeta_6 = \zeta_7 = 0.02$. The analysis period was 3.0 s with a time increment of 0.00025 s.

Transverse displacements were measured with 21 sensors at equidistant nodes shown numbered in Figure 1. Gaussian noise was added to the sensors, so that each sensor had an equal signal-to-noise ratio (SNR) of 15 dB, which is relatively low. The sensor noise variance was assumed to be known.

The modes of the same FE model were used for analytical virtual sensing. Ignoring model errors is actually unrealistic, but made it possible to study uncertainty reduction of different algorithms influenced by measurement error only.

Four different virtual sensing algorithms were studied:
1. Analytical VS using OLS estimation directly from the noisy measurement data.
2. Analytical VS using WLS estimation from the noisy measurement data.
3. Combined VS using OLS estimation from the empirical virtual sensors.
4. Combined VS using WLS estimation from the empirical virtual sensors.

The SNR of the actual measurements and the empirical estimates are plotted in Figure 2. It can be seen that noise could be considerably reduced from all sensors using empirical estimation.
Strain estimates with the exact values in the middle of element 52 (left column, elevation 3.2 m from the ground, see Figure 1) are plotted in Figure 3. It can be seen that combined VS resulted in considerable noise reduction compared to analytical VS. Moreover, WLS resulted in better estimates than OLS.

Figure 4 shows the SNR of all virtual displacement, rotation, and strain sensors, respectively. Combined VS using WLS outperformed the other three algorithms. Analytical VS using OLS resulted in the least accurate estimates.

The ranking of different algorithms at all 526 virtual sensors (175 displacements, 175 rotations, and 176 strains) is shown in Table 1. The table shows the number of virtual sensors ranked according to the accuracy of different algorithms. It can be seen that combined VS with WLS outperformed the other algorithms achieving all first places. Analytical VS with OLS was the most inaccurate for all virtual sensors.

The grand average of the SNR of all 526 virtual sensors using different algorithms is shown in Table 2. It can be seen that combined VS outperformed analytical VS. Combined VS using WLS had clearly the best performance. Also, combined VS using OLS performed well due to accurate empirical estimates $\hat{\gamma}_u$ (Equation 4).

### Table 1: The number of virtual sensors ranked according to accuracy of different algorithms.

<table>
<thead>
<tr>
<th>Ranking</th>
<th>Combined, OLS</th>
<th>Analytical, OLS</th>
<th>Combined, WLS</th>
<th>Analytical, WLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>0</td>
<td>0</td>
<td>526</td>
<td>0</td>
</tr>
<tr>
<td>2.</td>
<td>490</td>
<td>0</td>
<td>0</td>
<td>36</td>
</tr>
<tr>
<td>3.</td>
<td>36</td>
<td>0</td>
<td>0</td>
<td>490</td>
</tr>
<tr>
<td>4.</td>
<td>0</td>
<td>526</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

### Table 2: The grand average of the SNR of all virtual sensors using different algorithms.

<table>
<thead>
<tr>
<th>Combined, OLS</th>
<th>Analytical, OLS</th>
<th>Combined, WLS</th>
<th>Analytical, WLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>20.35</td>
<td>11.42</td>
<td>21.68</td>
<td>16.40</td>
</tr>
</tbody>
</table>

Figure 2: Signal-to-noise ratio (SNR) in each sensor. Measurement data (red) and empirical estimates (blue).
Figure 3: A detail of the virtual strain in the middle of element 52 using different algorithms. The red curve is the true strain.

Figure 4: Displacement, rotation, and strain SNR of each sensor using four different algorithms.
4. CONCLUSION

Combined empirical and analytical virtual sensing was introduced to estimate full-field dynamic response. The accuracy of combined VS was higher than that of analytical VS. Combined VS with WLS outperformed the other algorithms. WLS estimation produced better estimates than OLS, but the variance of the measurement error must be known.

If a large sensor network is available, all MMSE estimates are very accurate and combined VS using OLS and WLS would result in almost equal accuracy. OLS would then be preferred, because noise estimation is not needed.

Some important issues in virtual sensing were left outside this paper, but will be presented in the future: different sensor networks and noise models, different structures, and noise estimation. Also, model error was ignored in this study, and it was left for further studies. Often, vibrations are measured with accelerometers. When accelerations are used as physical measurements, they must be integrated twice to obtain displacements. Integration will be another source of error, but it was not studied in this paper. Optimal sensor replacement for virtual sensing is another subject for a future study. Experimental validation was also left for the future.

REFERENCES


