Damage identification in longitudinally vibrating rods based on quasi-isospectral operators

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**Abstract**
The paper proposes a new method for damage identification in longitudinally vibrating rods using natural frequency data. We assume that the damage reflects into a reduction of the axial stiffness without changing the mass density of the rod. The diagnostic procedure is of constructive type and it is based on the determination of the axial stiffness coefficient such that the first $N$ natural frequencies coincide with the measured values for the damaged rod. Main mathematical tool is the explicit construction of a suitable class of quasi-isospectral Sturm-Liouville operators. The analysis is applied to symmetric rods under supported end conditions. The analytical reconstruction was numerically implemented and tested on an extended series of damaged scenarios. Numerical results show that the accuracy of identification increases in case of small concentrated damages located within the central part of the rod.

1. **INTRODUCTION**

In this paper we consider the problem of identifying damage in a longitudinally vibrating rod having symmetric profile by using a finite number of natural frequencies. The free vibrations of a rod of length $L$ are governed by the Sturm-Liouville eigenvalue problem

\[
(pu')' + \lambda \rho u = 0, \quad \text{in } (0,L),
\]

where $(\lambda, u)$ is an eigenpair, and $p$, $\rho$ are the axial stiffness and the mass density, respectively. It is assumed that the occurrence of structural damage reflects into a reduction of the stiffness coefficient $p$, leaving the mass density $\rho$ unchanged. Under the assumption that the damage is symmetric with respect to the mid-point of the rod and under Dirichlet end conditions, we reconstruct the stiffness coefficient $p$ such that the damaged rod has the prescribed/measured values of the first lower eigenfrequencies.

The reconstruction method is based on the determination of quasi-isospectral operators of the type (1), with coefficient $\rho$ prescribed and coinciding with that of the undamaged rod. We recall that an isospectral rod is a rod which has the same spectrum of a given rod under a specified set of end conditions, with the exception of a single eigenvalue which is free to move in a prescribed interval. The explicit determination of quasi-isospectral rod operators of the type (1) with $p = \rho$ has been recently used in \cite{1} to construct axially vibrating rods having prescribed values of a finite number of lower natural frequencies. The rods and their normal modes can be constructed explicitly by means of closed-form expressions. The reconstruction procedure adopted in this paper is a non-trivial extension of a method we have recently used in \cite{2} to reconstruct blockages in ducts and horns, e.g., for which $p = \rho$. The procedure needs the specification of an initial rod (the undamaged one, in the present case) whose eigenvalues must be not far from those of the damaged rod.

The analytical procedure has been numerically implemented to test its effectiveness as diagnostic method. In particular, the results of a preliminary series of simulations performed on an initially uniform rod simply supported at the ends with single or multiple symmetric damages are presented and discussed.
2. FORMULATION OF THE PROBLEM

The free longitudinal vibration of a thin straight undamaged rod, having unit length and with both the ends supported, is governed by the eigenvalue problem

\[
\begin{align*}
\left\{ \begin{array}{l}
(p v')' + \hat{\lambda} \hat{p} v = 0, \\
v(0) = 0 = v(1).
\end{array} \right. & \quad \text{in } (0, 1), \\
\end{align*}
\]

Here, \( v = v(x) \) is the longitudinal displacement of the cross-section of abscissa \( x \); \( \sqrt{\hat{\lambda}} \) is the radian frequency of vibration; \( \hat{p} = \hat{p}(x) \) and \( \hat{\rho} = \hat{\rho}(x) \) are the axial stiffness and the mass density of the rod, respectively. We shall assume throughout that \( \hat{p} \) and \( \hat{\rho} \) are regular, positive functions of \( x \) in \([0, 1]\). Under the above assumptions, the problem (2)–(4) has a countable sequence of eigenvalues \( \{\lambda_m\}_{m=1}^\infty \), with \( 0 < \hat{\lambda}_1 < \hat{\lambda}_2 < \cdots \) and \( \lim_{m \to \infty} \hat{\lambda}_m = \infty \).

Let us assume that a structural damage occurs inside the rod and that the damage reflects into a modification (e.g., reduction) of the axial stiffness coefficient without affecting the mass density of the rod. Therefore, the eigenvalue problem for the damaged rod consists in finding the non-trivial solutions to

\[
\begin{align*}
\left\{ \begin{array}{l}
(p v')' + \lambda \hat{p} v = 0, \\
v(0) = 0 = v(1),
\end{array} \right. & \quad \text{in } (0, 1),
\end{align*}
\]

where the axial stiffness of the damaged rod \( p = p(x) \) is a regular positive function in \([0, 1]\). The eigenvalues \( \{\lambda_m\}_{m=1}^N \) of (4)–(5) are such that \( \hat{\lambda}_{m-1} \leq \lambda_m \leq \hat{\lambda}_m \) for every \( m \geq 1 \), with \( \hat{\lambda}_0 = 0 \).

Our inverse problem can be formulated as follows: given the undamaged configuration of the rod, we wish to determine the axial stiffness coefficient \( p = p(x) \) from the knowledge of the first \( N \) eigenvalues \( \{\lambda_m\}_{m=1}^N \) of the damaged rod. In order to avoid well-known non-uniqueness of the reconstruction, we shall assume that the undamaged and the damaged rod are symmetric with respect to the mid-point of the rod axis, that is

\[
\hat{p}(x) = \hat{p}(1 - x), \quad p(x) = p(1 - x), \quad \hat{\rho}(x) = \hat{\rho}(1 - x) \quad \text{in } [0, 1].
\]

3. THE RECONSTRUCTION PROCEDURE

The main steps of the reconstruction procedure are described in the sequel.

\textbf{STEP} 1. Under the above assumptions on the coefficients, let us consider the free vibration problem for the undamaged rod \( (\hat{p}, \hat{\rho}) \):

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{d}{ds}\left( \hat{p}(y) \frac{dy}{ds} \right) + \hat{\lambda} \hat{p}(y) v(y) = 0, \\
v(0) = 0 = v(1).
\end{array} \right. & \quad y \in (0, 1),
\end{align*}
\]

We apply the Liouville transformation

\[
x = \psi(y) = \frac{1}{M} \int_0^y \sqrt{\frac{\hat{p}(s)}{\hat{\rho}(s)}} ds, \quad y \in [0, 1], \quad \hat{M} = \int_0^1 \sqrt{\frac{\hat{p}(s)}{\hat{\rho}(s)}} ds,
\]

\[
\hat{A}(x) = (\hat{\rho}(y) \hat{p}(y))^{1/2}, \quad u(x) = v(y),
\]

\[
\text{to reduce problem (7)–(8) to the \textit{impedance-type} eigenvalue problem}
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{d}{dx}\left( \hat{A}(x) \frac{du(x)}{dx} \right) + \hat{\lambda} \hat{M}^2 \hat{A}(x) u(x) = 0, \\
u(0) = 0 = u(1),
\end{array} \right. & \quad x \in (0, 1),
\end{align*}
\]
where $\hat{A} \in C^2([0, 1])$ is an even strictly positive function, that is, $\hat{A}(x) = \hat{A}(1 - x)$ and $\hat{A}(x) \geq \gamma_0 > 0$ in $[0, 1]$.

**Step 2.** We adapt the constructive procedure presented in [2] to determine a strictly positive coefficient $A = A(x)$, with $A(x) \in C^2([0, 1])$ and $A(x) = A(1 - x)$, such that the eigenvalue problem

$$
\left\{ \begin{array}{l}
\frac{d}{dz} \left( A(x) \frac{du(x)}{dz} \right) + \lambda \hat{M}^2 A(x) u(x) = 0, \quad x \in (0, 1), \\
u(0) = 0 = u(1),
\end{array} \right.
$$

(13)

has exactly the first $N$ eigenvalues $\{\lambda_m\}_{m=1}^N$ of the damaged rod.

The procedure has been fully described in [2]. Referring to this paper for the details, here, for the sake of completeness, we sketch the main idea of the method. Starting from the Dirichlet impedance problem (11)–(12) with coefficient $\hat{A}(x)$, the key point of the method is the explicit determination of a new (positive and smooth) coefficient $A^*(x) = A^*(1 - x)$ such that the corresponding Dirichlet impedance problem has exactly the same spectrum as the given coefficient $\hat{A}(x)$, with the exception of a single eigenvalue which is free to move in a prescribed interval. It can be shown that this analysis is based on a double application of a Darboux lemma [3, 4] to obtain an impedance coefficient quasi-isospectral to the initial coefficient $\hat{A}(x)$. Once a recipe for the construction of a quasi-isospectral impedance operator is available, by using repeatedly the procedure, after $N$ steps one can determine a positive, smooth and even coefficient $A(x)$ with the first $N$ given (e.g., measured) Dirichlet eigenvalues $\{\lambda_m\}_{m=1}^N$. At every step of the procedure, the impedance coefficient can be constructed explicitly by means of closed-form expressions. Finally, it should be noted that a multiplicative constant on $A(x)$ has no effect on the eigenvalues. Therefore, in order to have uniqueness of the reconstruction (at least as $N \to \infty$), we will specify in the next step an additional - scalar - information on $p(x)$, precisely the value $\int_0^1 p(x) dx$.

**Step 3.** The Liouville transformation introduced in Step 1 is reversed to transform back the eigenvalue problem (13)–(14) into the eigenvalue problem for a supported rod with (unchanged) mass density $\hat{\rho}(x)$ and (positive, smooth and even) axial stiffness $p(x)$, such that the first $N$ eigenvalues $\{\lambda_m\}_{m=1}^N$ of $(p(x), \hat{\rho}(x))$ take the prescribed values assigned/measured for the damaged rod. More precisely, by assuming

$$
x = \phi(z) = \frac{1}{Q} \int_0^z f(s) ds, \quad z \in [0, 1],
$$

(15)

$$
f(s) = \sqrt{\frac{\rho(s)}{p(s)}}, \quad f(0) = f(1) = 0, \quad Q = \int_0^1 f(s) ds,
$$

(16)

$$
A(x) = (\rho(z) p(z))^{\frac{1}{2}}, \quad u(x) = v(z),
$$

(17)

the eigenvalue problem (13)–(14) becomes

$$
\left\{ \begin{array}{l}
\frac{d}{dz} \left( Q p(z) \frac{dv(z)}{dz} \right) + \lambda \hat{M}^2 \rho(z) v(z) = 0, \quad z \in (0, 1), \\
v(0) = 0 = v(1).
\end{array} \right.
$$

(18)

(19)

If we define

$$
\hat{\rho}(z) = \frac{\hat{M}^2}{Q} \rho(z), \quad z \in [0, 1],
$$

(20)

then it turns out that the axial stiffness coefficient is given by

$$
Q p(z) = \frac{\hat{\rho}(z)}{M^2 (\eta'(z))^2}, \quad z \in [0, 1],
$$

(21)

where the solution $\eta = \eta(z)$ of the Cauchy problem
\[
\begin{align*}
\eta'(z) &= \frac{\rho(z)}{M^2 A(\eta(z))}, \quad z \in \left(\frac{1}{2}, 1\right), \\
\eta\left(\frac{1}{2}\right) &= \frac{1}{2},
\end{align*}
\]

(22)

is extended to the whole interval \([0, 1]\) as follows

\[
\eta(z) = \eta(1-z), \quad z \in [0, 1].
\]

(24)

Note that the function \(p(z)\) is a positive, smooth and even function in \([0, 1]\). Finally, the supported rod in (18)–(19), having the same mass density \(\hat{\rho}\) of the undamaged rod in (7)–(13) and with axial stiffness \(Qp(z)\), is the wished damaged rod.

4. NUMERICAL RESULTS

In this section we shall present some preliminary results of the numerical implementation of the reconstruction procedure illustrated in previous sections. The results refer to an initially uniform rod simply supported at the ends with \(L = 1\), \(\hat{\rho}(x) = \hat{\rho}(x) = 1\) in \([0, 1]\). The axial stiffness of the damaged rod \(p(x)\) is assumed to have single or multiple localized damages, whose typical profile is modelled by a fifth-degree Bézier curve (see [5]), see Figure 1.

The natural frequencies of the damaged \(p(x)\)-rod were determined by using a numerical tool based on B-spline finite elements on a very refined mesh consisting of 12800 elements, see [6]. The numerical procedure described in [2] was used to solve Step 2. The Cauchy problem (22)–(23) was solved by means of a single-step method based on a trapezium rule. We notice that the finite element mesh used for the reconstruction is more sparse than that adopted for the solution of the direct eigenvalue problem, i.e., 1600 finite elements instead of 12800.

Some results are shown in Figures 2–6. Black and red curves represent the exact and the reconstructed profile, respectively. At the top of each sub-figure, the eigenfrequency percentage shifts \(\delta_m\) induced by the damage on the uniform rod are represented. The differences \(\rho_m\) between computed (after the reconstruction) and target eigenfrequency values are shown on the bottom. All the data have been obtained by imposing the first \(N = 20\) natural frequencies, with the exception of the results shown in Figures 7–8, which were obtained using \(N = 40\).

In particular, Figures 2–3 and Figures 4–6 consider the effect of increasing the severity (e.g., increasing \(e\)) and the extension (e.g., increasing \(c\)) of the damage, respectively. It can be seen that the actual profile is reconstructed with good accuracy and damages are well-individuated, both for the position and severity. However, it should be noticed that identification of large damages contains some spurious fluctuation of the axial stiffness coefficient \(p(x)\) near the ends of the rod.
5. CONCLUSIONS

We have presented a new method for the identification of damages in a longitudinally vibrating rod by using natural frequency data. The procedure is based on the explicit determination of the axial stiffness of the rod such that the first lower natural frequencies coincide with the measured values for the damaged rod. A preliminary series of numerical simulations shows the effectiveness of the diagnostic method in presence of small concentrated damages located far enough from the ends of the beam. Further analysis is needed to improve the quality of the identification, particularly to reduce the wavy behavior of the reconstructed profile near the ends of the rod, and to test the stability of the method to errors on the frequency data. These issues will be the object of future research.

REFERENCES


Figure 3: Reconstruction test for a damage defined by $b = 0.24$, $c = 0.025$, $e = 0.4$ and $N = 20$.

Figure 4: Reconstruction test for a damage defined by $b = 0.24$, $c = 0.05$, $e = 0.2$ and $N = 20$. 
Figure 5: Reconstruction test for a damage defined by $b = 0.24$, $c = 0.05$, $e = 0.4$ and $N = 20$.

Figure 6: Reconstruction test for a damage defined by $b = 0.24$, $c = 0.075$, $e = 0.2$ and $N = 20$. 
Figure 7: Reconstruction test for a damage defined by $b = 0.24$, $c = 0.025$, $e = 0.2$ and $N = 40$.

Figure 8: Reconstruction test for a damage defined by $b = 0.24$, $c = 0.025$, $e = 0.4$ and $N = 40$. 
Figure 9: Three localized damages, with $b = 0.24$, $c = 0.025$, $e = 0.4$ and $N = 20$.

Figure 10: Four localized damages, with $b = 0.24$, $c = 0.025$, $e = 0.4$ and $N = 20$. 